Generalized-hollow lifting\(_g\) modules

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Abstract

Let \( R \) be any ring with identity, and let \( M \) be a unitary left \( R \)-module. A submodule \( K \) of \( M \) is called generalized coessential submodule of \( N \) in \( M \), if \( (N)/K \subseteq \text{Rad}_g(M/K) \). A module \( M \) is called generalized hollow-lifting module, if every submodule \( N \) of \( M \) with \( M/N \) is a \( G \)-hollow module, has a generalized coessential submodule of \( N \) in \( M \) that is a direct summand of \( M \). In this paper, we study some properties of this type of modules.

Keywords: generalized coessential submodule, generalized strong supplement submodule, generalized hollow-lifting \(_g\) module.

1. Introduction

Throughout this paper \( R \) is a ring with identity, and every \( R \)-module is a unitary left \( R \)-module, \( N \subseteq M \) denotes \( N \) is a submodule of \( M \). Let \( M \) be an \( R \)-module, and let \( N \subseteq M \), \( N \) is called essential submodule of \( M \) (denoted by \( N \subseteq_e M \)) if every nonzero submodule \( B \) of \( M \), we have \( B \cap A \neq 0 \) [1]. A submodule \( N \) of \( M \) is called small submodule of \( M \) (denoted by \( N \ll M \)), if for every \( K \subseteq M \), \( M=N+K \) implies \( K=M \) [2]. \( \text{Rad}(M) \) is the sum of all small submodules of \( M \) [2]. A submodule \( N \) of \( M \) is called generalized small submodule of \( M \) (for short, \( G \)-small) (denoted by \( N \ll_G M \)), if for every \( K \subseteq_e M \), \( M= N+K \) implies \( K=M \) [3]. \( \text{Rad}_g(M) \) is the sum of all \( G \)-small of \( M \)[3]. It clear that \( \text{Rad}(M) \subseteq \text{Rad}_g(M) \), but the converse is not true in general. A nonzero module \( M \) is called generalized-hollow (for short, \( G \)-hollow), if every proper submodule of \( M \) \( G \)-small (in [4], it is denoted by \( e \)-hollow). A Submodule \( K \) of \( M \) is called coessential submodule of \( N \) in \( M \) (denoted by \( K \ll_e M \)), if \( N/K \ll M/K \). A module \( M \) is called lifting module or satisfies (D1) if for every submodule \( N \) of \( M \) there exists a direct summand \( K \) of \( M \) such that \( M = K \uplus K' \), \( K \subseteq N \), \( K' \subseteq M \) and \( N \cap K' \ll M \)[5]. \( M \) is called hollow lifting, if for every submodule \( N \) of \( M \) with \( M/N \)is hollow has a coessential submodule in \( M \) that is a direct summand of \( M \), [6]. Clearly every lifting module is hollow lifting , while the converse does not hold in general , see [6]. A submodule \( K \) of \( M \) is called \( G \)-

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coessential submodule of $N$ in $M$ (denoted by $K \subseteq \text{Gce}N$), if $N/K \ll (G) M/K$,[7]. An $R$-module $M$ is called generalized lifting or satisfies (GD1), if for every submodule $N$ of $M$, there exists a direct summand $K$ of $M$, such that $K \subseteq \text{Gce}N$ in $M$.[4]. It is clear that every lifting module is a generalized lifting module.

An $R$-module $M$ is called a generalized hollow lifting module (for short, $G$-hollow lifting module), if for every submodule $N$ of $M$, with $M/N$ is hollow module, $N$ has a generalized coessential submodule of $M$ that is a direct summand of $M$. [7]

In this paper we introduce a generalized hollow lifting module as a generalization of generalized hollow lifting module.

Let $N$, $K \subseteq M$, $N$ is called supplement of $K$ in $M$ if $M=N+K$ and $N \cap K \ll N$.[2] and $N$ is called strong supplement of $K$ if $N$ is a supplement of $K$ in $M$ and $N \cap K$ is a direct summand of $K$.[3].

We introduce $G$-strong supplement submodule, let $N$, $K \subseteq M$ we called $K$ is $G$-strong supplement of $N$ in $M$ if $M=N+K$, $N \cap K \subseteq \text{Rad}M$ and $N \cap K$ is a direct summand of $N$.

In fact, we prove for an indecomposable module $M$, $M$ is $G$-hollow-lifting module if and only if $M$ is $G$-hollow or else $M$ has no $G$-hollow factor module. We also prove that for $N \subseteq M$, $N$ has a generalized strong supplement in $M$ if and only if $N$ has a generalized coessential submodule that is a direct summand of $M$. therefore $M$ is a $G$-hollow-lifting module if and only if for every submodule $N$ of $M$, with $M/N$ is $G$-hollow has a generalized strong supplement in $M$.

In section three, we prove that for fully invariant submodule of $M$, if $M$ is $G$-hollow-lifting module, then $M/N$ is a $G$-hollow-lifting module. In fact, we give sufficient condition for direct sum of two $G$-hollow lifting module to be $G$-hollow lifting. We prove if $M=M_1 \oplus M_2$ is a duo module, then $M$ is a $G$-hollow-lifting module, if and only if $M_1$, and $M_2$ are $G$-hollow-lifting modules.

2. Some properties of $G$-hollow lifting modules

In this section, we introduce $G$-hollow lifting module as a generalization of hollow lifting module, and study some properties of this type of modules.

**Definition 2.1**[7]: A submodule $K$ of $M$ is called generalized coessential submodule of $N$ in $M$ denoted by

$$K \subseteq \text{Gce}N, \frac{N}{K} \subseteq \text{Rad}M \frac{K}{K}.$$  

It is clear that, if $K$ is coessential submodule of $N$ in $M$, then $K$ is generalized coessential submodule of $N$ in $M$. However the converse in general is not true, for example $0 \subseteq Q$ as $Z$-module, but $0$ is not coessential of $Q$.

**Definition 2.2**[4]: An $R$-module $M$ is called generalized lifting or satisfies (GD1), if for every submodule $N$ of $M$, there exists a direct summand $K$ of $M$, such that $K \subseteq \text{Gce}N$ in $M$.

It is clear that every lifting module is a generalized lifting module. An $R$-module $M$ is called hollow lifting, if every submodule $N$ of $M$ such that $K \subseteq N$-hollow has a coessential submodule that is a direct summand of $M$.[6].

It is known that $\text{Rad}(M) \subseteq \text{Rad}_g(M)$.[8].

The following gives the properties of $\text{Rad}_g(M)$ which appeared in [8].

**Lemma 2.3**: The following assertions are holds:

1. If $M$ be an $R$-module, then $Rm \ll M$ for every $m \in \text{Rad}(M)$.
2. If $f:M \rightarrow N$ is an $R$-module homomorphism, then $f(\text{Rad}_g(M)) \subseteq \text{Rad}_g(N)$.
3. If $N \subseteq M$, then $\text{Rad}_g(N) \subseteq \text{Rad}_g(M)$.
4. If $K,L \subseteq M$, then $\text{Rad}_g(K)+\text{Rad}_g(L) \subseteq \text{Rad}_g(K+L)$.
5. If $K,L \subseteq M$, then $\text{Rad}_g(K+L) = \text{Rad}_g(K)+\text{Rad}_g(L)$.
6. If $M=\oplus_{i \in I} M_i$, then $\text{Rad}_g(M) = \oplus_{i \in I} \text{Rad}_g(M_i)$.

**Lemma 2.4**: Let $N$ be a direct summand submodule of $M$. Then $\text{Rad}_g(N) = \text{Rad}_g(M) \cap N$.

**Proof**: See [8].

As a generalization of generalized of hollow lifting module we introduce the following:

**Definition 2.5**: An $R$-module $M$ is called $G$-hollow lifting module, if for every submodule $N$ of $M$ with $M/N$ is $G$-hollow has a $G$-coessential submodule in $M$ that is a direct summand of $M$.

**Examples and Remarks 2.6**:
1- \( Z_4 \) as \( Z \)-module is G-hollow \( \text{lifting} \_g \) module.

2- \( M=Z_{12} \) as \( Z \)-module is not G-hollow \( \text{lifting} \_g \) module, since let \( N=\langle 2 \rangle \) and \( K=\langle 4 \rangle \) is a direct summand of \( M \). 

Proposition 2.7: Let \( M \) be a G-hollow \( \text{lifting} \_g \) module , then every submodule \( N \) of \( M \) such that \( \frac{M}{N} \) is G-hollow, can be written as \( N = K \bigoplus L \), where \( K \) is a direct summand of \( M \) and \( N \cap L \subseteq \text{Rad}_g(M) \).

Proof: Let \( N \subseteq M \), with \( \frac{M}{N} \) is G-hollow, since \( M \) is a G-hollow \( \text{lifting} \_g \) module, then \( \exists K \subseteq M \), \( K \subseteq N \) and \( \frac{N}{K} \subseteq \text{Rad}_g\left(\frac{M}{K}\right) \), let \( L \subseteq M \) with \( M = K \bigoplus L \) then \( N = K \bigoplus (L \cap N) \). Now \( \frac{N}{K} = \frac{(K \bigoplus (L \cap N))}{K} = \frac{N \cap L}{K \cap (N \cap L)} \) Thus \( N \cap L \subseteq \text{Rad}_g\left(\frac{M}{K}\right) \) since \( K \) is a strong supplement submodule in \( M \).

Proposition 2.8: Let \( M_1 \) and \( M_2 \) be \( G \)-hollow modules, if \( M = M_1 \bigoplus M_2 \) then the following are equivalent:

1. \( M \) is G-hollow \( \text{lifting} \_g \).
2. \( M \) is G-lifting.

Proof: \( 1 \rightarrow 2 \) Let \( N \subseteq M \), let \( \pi_1 : M \rightarrow M_1 \) and \( \pi_2 : M \rightarrow M_2 \). If \( \pi_1 (N) \neq M_1 \) and \( \pi_1 (N) \neq M_2 \), then \( \pi_1 (N) = M_1 \) and \( \pi_2 (N) = M_2 \). Thus \( \pi_1 (N) \bigoplus \pi_2 (N) = M_1 \bigoplus M_2 \) [9].

Now let \( n \in N \), then \( n = m_1 + m_2 \), where \( m_1 \in M_1 \) and \( m_2 \in M_2 \). \( \pi_1 (n) = \pi_1 (m_1 + m_2) = m_1 \) and \( \pi_2 (n) = \pi_2 (m_1 + m_2) = m_2 \), thus \( n = \pi_1 (n) + \pi_2 (n) \) this implies that \( N \subseteq \pi_1 (N) \bigoplus \pi_2 (N) \) therefore \( N \subseteq M \). Assume that \( \pi_1 (N) = M_1 \) then \( M = M_1 + M_2 \), thus \( M/N = M_1 + M_2 \) and \( N \) is G-hollow, hence \( M = M_1 + M_2 \) is G-hollow, since \( M_1 + M_2 \) is G-hollow, therefore \( \exists K \subseteq M \) such that \( N \subseteq \text{Rad}_g(M) \), hence \( M \) is a generalized lifting.

Remark 2.9: It is clear that every module has no hollow factor module is a G-hollow \( \text{lifting} \_g \) module . However, if \( M \) is indecomposable we have the following:

Proposition 2.10: Let \( M \) be an indecomposable module, then the following are equivalent:

1. \( M \) is G-hollow \( \text{lifting} \_g \) module.
2. \( M \) is G-hollow or else \( M \) has no G-hollow factor module.

Proof: \( 1 \rightarrow 2 \) Suppose that \( M \) has a G-hollow factor module, then \( \exists N \subseteq M \), such that \( \frac{M}{N} \) is G-hollow .

Since \( M \) is G-hollow \( \text{lifting} \_g \) module., then \( \exists K \subseteq M \), \( K \subseteq M \) such that \( \frac{N}{K} \subseteq \text{Rad}_g(M) \). But \( M \) is indecomposable, then \( K = 0 \) and hence \( N \subseteq \text{Rad}_g(M) \).

\( 2 \rightarrow 1 \) Clear.

Let \( R \) be any ring, and \( M \) is an \( R \)-module. Let \( N, K \) be two submodules of \( M, K \) is called strong supplement of \( N \) in \( M \), if \( K \) is a supplement of \( N \) in \( M \), and \( K \cap N \) is a direct summand of \( N \).[3]

As a generalization of strong supplement submodule, we introduce the following:

Definition 2.11: Let \( N, K \) be submodules of \( M \). \( K \) is called a generalized strong supplement of \( N \) (for short G-strong supplement of \( N \)), if \( M = N + K \) with \( K \cap N \subseteq \text{Rad}_g(K) \) and \( K \cap N \subseteq \text{Rad}_g(N) \).

It is clear that if \( K \) is strong supplement submodule in \( M \), then \( K \) is G-strong supplement submodule, but the converse in general is not true , for example: consider \( Z_{12} \) as \( Z \)-module , let \( N = \{ 0, 4, 8 \} \), it is clear that \( N \) is G-strong supplement since there exist a direct summand \( 0 \) of \( M \), \( N \equiv G \), but \( N \) not small in \( M \).

Remark 2.12: In semisimple modules, every submodule is G-strong supplement.

Proposition 2.13: Let \( N \subseteq M \), then the following are equivalent:

1. \( N \) has a G-strong supplement in \( M \).
2. \( N \) has a G-coessential submodule that is a direct summand of \( M \).

Proof: \( 1 \rightarrow 2 \) Let \( K \) be a G-strong supplement of \( N \) in \( M \), then \( M = N + K \), \( N \cap K \subseteq \text{Rad}_g(M) \) and \( N \cap K \subseteq \text{Rad}_g(N) \), hence \( N \subseteq N \cap K \) such that \( (N \cap K) \bigoplus L = N \), then \( M = L \bigoplus K \). Now \( \frac{N}{L} = \frac{(N \cap K) \bigoplus L}{L} \subseteq \frac{\text{Rad}_g(M) \bigoplus L}{L} \subseteq \frac{\text{Rad}_g(M)}{L} \).
Let $N \subseteq M$ such that $M=K \oplus L$ and $K$ is a direct summand of $M$, hence $M=K \oplus L$ for $L \subseteq M$. Thus $N=K \cap (N \cap L)$ and $N \cap L$ is a direct summand of $N$.  

Now $N \subseteq \frac{K+(N \cap L)}{K} = \frac{(N \cap L)}{L} = N \cap L$. 

But $N \subseteq \text{Rad}_g \left( \frac{M}{K} \right)$, hence $N \cap L \subseteq \text{Rad}_g \left( \frac{M}{K} \right)$. Thus $N$ has a $G$-strong supplement in $M$. 

**Corollary 2.14**: Let $M$ be any $R$-module, then the following are equivalent:

1. $M$ is a $G$-hollow $\text{lifting}_g$ module.
2. Every submodule $N$ of $M$, with $M/\mathbb{N}$ is $G$-hollow, has a $G$-strong supplement in $M$. 

**Proposition 2.15**: Let $M$ be a $G$-hollow module, then the following are equivalent:

1. $M$ is a $G$-hollow $\text{lifting}_g$ module.
2. $M$ is a $G$-lifting module. 

**Proof**: $1 \rightarrow 2$ by [4], for any $N \subseteq M, \frac{M}{N}$ is $G$-hollow and by (1) $M$ is $G$-lifting. 

$2 \rightarrow 1$ Clear.

**3. The direct sum of $G$-hollow $\text{lifting}_g$ module**

In this section we study the quotient and the direct sum of $G$-hollow $\text{lifting}_g$ module, we prove under certain condition the quotient and the direct summand of $G$-hollow $\text{lifting}_g$ module is $G$-hollow $\text{lifting}_g$ module.

**Remark 3.1**: The quotient module of $G$-hollow $\text{lifting}_g$ module needn’t be $G$-hollow $\text{lifting}_g$, the following example shows:

**Example 3.2**: Consider the $Z$-module $M = \frac{Z}{4Z} \oplus \frac{Z}{8Z}$, let $N = \frac{Z}{4Z} \oplus \frac{Z}{8Z} < 0$, clearly that $M$ is $G$-hollow $\text{lifting}_g$ module, since it is lifting but $\frac{M}{N}$ is not, since $\frac{M}{N} = \frac{\frac{Z}{4Z} \oplus \frac{Z}{8Z}}{\frac{Z}{4Z} \oplus \frac{Z}{8Z}} < 0$. Then $\frac{M}{N} = \frac{Z}{4Z} \oplus \frac{Z}{8Z}$ which is not $G$-hollow $\text{lifting}_g$. 

Recall that a submodule $N$ of $M$ is called fully invariant if $f(N) \subseteq N$ for every $f \in \text{End}(M)$, and an $R$-module $M$ is called duo module, if every submodule of $M$ is fully invariant.[10].

**Proposition 3.3**: Let $M$ be any $R$-module, if $M$ is a $G$-hollow $\text{lifting}_g$ module, then $\frac{M}{N}$ is a $G$-hollow $\text{lifting}_g$ module, for every fully invariant submodule $N$ of $M$.

**Proof**: Let $N$ be a fully invariant submodule of $M$, and let $\frac{K}{N} \subseteq \frac{M}{N}$ such that $\frac{M}{N} \subseteq \frac{M}{K}$ is $G$-hollow. 

Since $M$ is $G$-hollow $\text{lifting}_g$, then $\exists L \subseteq \oplus M$, such that $L \subseteq K, \frac{K}{L} \subseteq \text{Rad}_g \left( \frac{M}{K} \right)$ and $M = K \oplus L$ for $K, L \subseteq M$, clearly $N \cap L \subseteq K$, then $\frac{L+N}{\mathbb{N}} \subseteq \frac{K}{N}$. Define $f: \frac{M}{L} \rightarrow \frac{M}{N+L}$ by $f(\frac{m+l}{N})=m+(L+N), \forall m \subseteq M$. It is clear that $f$ is an epimorphism, $f(\frac{K}{L}) \subseteq \text{Rad}_g \left( \frac{M}{N+L} \right)$, hence $K+(L+N) \subseteq \frac{M}{N+L}$, hence $\frac{K}{N+L} \subseteq \text{Rad}_g \left( \frac{M}{N+L} \right)$. 

Now $\frac{M}{N} \subseteq \frac{K_1+N}{N} \oplus \frac{L+N}{N}$, hence $L+N/\mathbb{N} \subseteq \frac{M}{N}$, thus $\frac{M}{N}$ is a $G$-hollow $\text{lifting}_g$ module. 

**Corollary 3.4**: The direct summand of duo $G$-hollow $\text{lifting}_g$ module is again $G$-hollow $\text{lifting}_g$ module. 

**Remark 3.5**: The direct sum of two $G$-hollow $\text{lifting}_g$ modules need not be a $G$-hollow lifting as the following example shows:

**Example 3.6**: The modules $Z_4$ and $Z_3$ as $Z$-module are $G$-hollow $\text{lifting}_g$ modules. 

While the module $Z_4 \oplus Z_3 \cong Z_{12}$ which is not $G$-hollow $\text{lifting}_g$ module. 

The following shows under certain condition the direct sum of two $G$-hollow $\text{lifting}_g$ is again $G$-hollow $\text{lifting}_g$ module. 

**Proposition 3.7**: Let $M$ be a $G$-hollow module such that $M = M_1 \oplus M_2$, if $M_1$ and $M_2$ are $G$-hollow $\text{lifting}_g$ modules, then $M$ is a $G$-hollow $\text{lifting}_g$ module.
Proof: Let \( N \subseteq M \) with \( \frac{M}{N} \) is G-hollow, then \( N \cap M = (N \cap M_1) \oplus (N \cap M_2) \) by [9]. Hence \( \frac{M}{N} \)
\( = \frac{M_2 \oplus M_3}{(N \cap M_1) \oplus (N \cap M_2)} \cong \frac{M_1}{N \cap M_1} \oplus \frac{M_2}{N \cap M_2} \), thus \( \frac{M}{N} = \frac{M_2}{N \cap M_2} \) is G-hollow, and similarly \( \frac{M_3}{N \cap M_1} = \frac{M_2}{N \cap M_2} \) is G-hollow.

Since \( M_1 \) and \( M_2 \) are G-hollow lifting module, then \( \exists k_1 \in \oplus M_1 \) with \( k_1 \subseteq N \cap M_1 \) and \( \frac{N \cap M_1}{K} \subseteq \text{Rad}_g \left( \frac{M_1}{K_1} \right) \), \( M_1 = K_1 \oplus L_1 \), \( L_1 \subseteq M_1 \) and \( \exists K_2 \subseteq \oplus M_2 \) with \( K_2 \subseteq N \cap M_2 \) and \( \frac{N \cap M_2}{K_2} \subseteq \text{Rad}_g \left( \frac{M_2}{K_2} \right) \), \( M_2 = K_2 \oplus L_2 \), \( L_2 \subseteq M_2 \). Thus \( K_1 + K_2 \subseteq (N \cap M_1) + (N \cap M_2) = N \) and \( K_1 + K_2 \oplus L_1 + K_2 = M_1 \oplus M_2 = M \). Thus \( K_1 \oplus K_2 \subseteq \oplus M \).

Now \( \frac{N}{K_1 + K_2} = \frac{(N \cap M_1) \oplus (N \cap M_2)}{K_1 + K_2} \subseteq \text{Rad}_g \left( \frac{M_1}{K_1} \right) + \text{Rad}_g \left( \frac{M_2}{K_2} \right) \subseteq \text{Rad}_g \left( \frac{M}{K_1 + K_2} \right) \). Then \( K_1 + K_2 \subseteq \text{G-cot} N \), and hence \( M \) is G-hollow lifting module.

Corollary 3.8: let \( M = M_1 \oplus M_2 \oplus \ldots \oplus M_n \) be a duo module if \( \forall i = 1, 2, \ldots, n, M_i \) is a G-hollow lifting module, then \( M \) is M is a G-hollow lifting module.

References