Certain Properties for Analytic Functions Associated with q-Ruscheweyh Differential Operator

Osamah N. Kassar*, Abdul Rahman S. Juma
Department of Mathematics, University of Anbar, Ramadi, Iraq

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Abstract
In this paper, by making use of the q-Ruscheweyh differential operator $R_q f(z)$, and the notion of the Janowski function, we study some subclasses of holomorphic functions. Moreover, we obtain some geometric characteristic such as coefficient estimates, radii of starlikeness, distortion theorem, close-to-convexity, convexity, extreme points, neighborhoods, and the integral mean inequalities of functions affiliated to these classes.

Keywords: Analytic functions, Subordination, q-Ruscheweyh derivative, Hadamard product, Univalent functions.

q–Ruscheweyh بعض الخصائص لدوال التحليلية المرتبطة بالمؤثر التفاضلي

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Introduction
Let $A$ represents the class of functions $f$ which are holomorphic functions in the unit disc $E^* = \{ z \in \mathbb{C} : |z| < 1 \}$ and of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad (z \in E^*) \quad (1)$$

The subclass of $A$ consisting of univalent functions is denoted by $S$. A function $f$ in $A$ is said to be starlike of order $\sigma (0 \leq \sigma < 1)$ in $E^*$ if this condition satisfies

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \sigma, \quad (Z \in E^*).$$

$S^*(\sigma)$ symbolize this class. In certain, for $(\sigma = 0)$, we obtain $S^*(0) = S^*$, the class of starlike functions. The class $C(\sigma), (0 \leq \sigma < 1)$ comprised of convex functions of order $\sigma$ can be expressed by the relation $f \in C(\sigma)$ if and only if $zf' \in S^*(\sigma)$.

*Email: os1989ama@uoaanbar.edu.iq
Let $f$ and $g$ be holomorphic functions such that both the subordination between $f$ and $g$ in $E^*$ are written as $f < g$ or $f(z) < g(z)$. In addition to that, we say that $f$ is subordinate to $g$ if there is a Schwarz function $w$ with $w(z) = 0, |w(z)| < 1, z \in E^*$, such that $f(z) = g(w(z))$ for all $z \in E^*$. Furthermore, if $g(z)$ is univalent in $E^*$, then we have the following equivalence:

$$f < g \text{ if and only if } f(0) = g(0) \text{ and } f(E^*) \subseteq g(E^*).$$

For some details, see earlier works [1,2,3]. By the application of the notion of subordination, Janowski provided the class $P[A,B]$. A given holomorphic function $h$ with $h(0) = 1$ is said to be in the class $P[A,B]$, if and only if the following condition satisfies:

$$h(z) < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1.$$  

Geometrically, the function $h(z) \in P[A,B]$ maps the unit disk $E^*$ onto the domain $E[A,B]$ defined by

$$E[A,B] = \left\{ w : \frac{1 - AB}{1 - B^2} \right\}. \quad (2)$$

This domain symbolizes an open circular disk centered on real axis with diameter end points $D_1 = \frac{1-A}{1-B}$ and $D_2 = \frac{1+A}{1+B}$ with $0 < D_1 < 1 < D_2$.

Consider $f, g \in A$. Then, the convolution * or Hadamard product of $f$ in $A$ and $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$ are defined as:

$$(f * g) = z + \sum_{m=2}^{\infty} a_m b_m z^m. \quad (z \in E^*).$$

Now, we define the Ruscheweyh derivative operator $\mathcal{R}^k$ as follows

$$\mathcal{R}^k f(z) = \frac{z}{(1-z)^{k+1}} * f(z).$$

Hence

$$\mathcal{R}^k f(z) = \frac{z}{(1-z)^{k+1}} f(z).$$

For $\in \mathbb{N}_0 = \{0,1,2,3,... \}, z \in E^*$, as previously described [4]. We briefly recover here the concept of q-operators, i.e., q-difference operator that takes a vital role in hyper geometric series, quantum physics, and operator theories. The usage of q-calculus was initiated by Jackson [5] (also see [6, 7]). For the applications of q-calculus in geometric function theory, one may refer to the papers of Mohamad and Darus [8], Mohamad and Sokol [9], and Purohit and Raina [10].

Next, we provide some fundamental definitions and results of $q$-calculus which we shall apply in our results. For more information, see earlier reports [10,11,12]. The application of q-calculus was initiated by JacksOn [13] (also see [14,15]) in geometric function theory.

Now, if $q \in (0,1)$ is fixed, then JacksOn explained the q-derivative and the q-integral of $f \in A$ as in the next step:

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad (q \in (0,1), z \in E^*). \quad (3)$$

and

$$\int_0^z f(t) \partial_q t = z(1-q) \sum_{m=0}^{\infty} q^m f(zq^m),$$

if that series converges.

It can simply be seen that for $m \in \mathbb{N}_0 = \{0,1,2,3,... \}$ and $z \in E^*$,

$$\partial_q \left\{ \sum_{m=1}^{\infty} a_m z^m \right\} = \sum_{m=1}^{\infty} [m,q]a_m z^{m-1},$$

where

$$[m,q] = \frac{1 - q^m}{1-q} = \sum_{i=1}^{m-1} q^i + 1, \quad [0,q] = 0. \quad (4)$$

2351
For every non-negative integer \(n\), the \(q\)-number shift factorial is defined by
\[
[m, q]! = \begin{cases} 
1, & m = 0 \\
[1, q][2, q][3, q] \ldots [m, q], & m \in \mathbb{N}. 
\end{cases}
\]

In addition, the \(q\)-generalized Pochhammer symbol for \(y > 0\) is defined as
\[
[m, q]! = \begin{cases} 
1, & m = 0 \\
[y, q][y + 1, q] \ldots [y + m - 1, q], & m \in \mathbb{N}. 
\end{cases}
\]

Let \(F\) be the function given as
\[
F_{k+1,q}(z) = z + \sum_{m=2}^{\infty} \frac{[k + 1, q]_{m-1}}{[m - 1, q]!} z^m. 
\tag{5}
\]

Now, the differential \(q\)-Ruscheweyh operator \(\mathcal{R}^k_q: \mathcal{A} \rightarrow \mathcal{A}\) of order \(k \in \mathbb{N}_0 = \{0,1,2,\ldots\}, q \in (0,1)\) and for \(f\) given by (1) is defined as
\[
\mathcal{R}^k_q f(z) = F_{k+1,q}(z) * f(z) = z + \sum_{m=2}^{\infty} \vartheta^k_q[m]a_m z^m, \tag{6}
\]
where \(\vartheta^k_q[m] = \frac{[k+1, q]_{m-1}}{[m-1, q]!}\) (for more details see a previous report [16]), and \(\mathcal{R}^k_q f(z) = f(z)\), also \(\mathcal{R}^k_q f(z) = z \vartheta_q f(z)\).

Equation (6) can be expressed as
\[
\mathcal{R}^k_q f(z) = \frac{z \vartheta_q^k \left( z^{-1} f(z) \right)}{[k, q]!}, \quad k \in \mathbb{N}_0.
\]

Since
\[
\lim_{q \to 1^{-}} F_{k+1,q}(z) = \frac{z}{(1-z)^{k+1}}, 
\]

it follows that
\[
\lim_{q \to 1^{-}} \mathcal{R}^k_q f(z) = \frac{z}{(1-z)^{k+1}} * f(z) = \mathcal{R}^k.
\]

**Definition 1.** Let \(\mathcal{J}_{k,j}(q, \beta, A, B)\) indicate the subclass of \(\mathcal{A}\) consisting of functions \(f\) of the form (1) and satisfy the following subordination condition,
\[
\left| \frac{\mathcal{R}^k_q f(z)}{\mathcal{R}^j_q f(z)} - \beta \right| < \frac{1 + Az}{1 + Bz},
\]
where \(-1 \leq B < A \leq 1, \beta \geq 0, k \in \mathbb{N}_0, f \in \mathbb{N}_0, k > j, q \in (0,1), z \in \mathbb{E}^*
\]

We note the following:
(i) For \(A = 1 + 2\alpha; B = -1; \beta = 0; k = 1\) and \(j = 0\), the class \(\mathcal{J}_{k,j}(q, \beta, A, B)\) reduces to the class \(S_q^{\ast}(\alpha)\) discussed by Agrawal and Sahoo [17].
(ii) For \(A = 1; B = -1; \beta = 0; k = 1\) and \(j = 0\), the class \(\mathcal{J}_{k,j}(q, \beta, A, B)\) reduces to the class \(S_q^{\ast}(A, B)\) discussed by Libera [19].
(iii) For \(q \to 1; \beta = 0; k = 1\) and \(j = 0\), the class \(\mathcal{J}_{k,j}(q, \beta, A, B)\) reduces to the class \(S_q^{\ast}(A, B)\) discussed by Janowski [20].
(iv) For \(q \to 1; \beta = 0; k = 1\) and \(j = 0\), the class \(\mathcal{J}_{k,j}(q, \beta, A, B)\) reduces to class \(S^{\ast}(A, B)\) discussed by Eker and Owa [22].
(v) For \(q \to 1; \beta = 0; k = 2\) and \(j = 1\), the class \(\mathcal{J}_{k,j}(q, \beta, A, B)\) reduces to the class \(K(A, B)\) discussed by Padmanabhan and Ganesan [21].
(vi) For \(q \to 1; B = -1\) and \(A = 1 - 2\alpha\), the class \(\mathcal{J}_{k,j}(q, \beta, A, B)\) reduces to class \(N_{k,j}(\alpha, \beta), (0 \leq \alpha < 1)\) discussed by Eker and Owa [22].
(vii) For \(k = 1; q \to 1; A = 1 - 2\alpha; B = -1\) and \(j = 0\), the class \(\mathcal{J}_{k,j}(q, \beta, A, B)\) reduces to the class \(U \mathcal{S}(\alpha, \beta)\) \((0 \leq \alpha < 1)\) discussed by Shams et al.[23].

**Definition 2.** Let \(\mathcal{T}\) represents the subclass of functions of \(\mathcal{A}\) of the form:
\[
f(z) = z - \sum_{m=2}^{\infty} a_m z^m, \quad a_m \geq 0. \tag{7}
\]
Further, we define the class $\mathcal{J}_{k,j}(q, \beta, A, B) = \mathcal{J}_{k,j}(q, \beta, A, B) \cap \mathcal{T}$.

For more details refer to an earlier work [24].

**Main Results**

In this part, we will prove our main results.

**Theorem 1.** A function $f$ of the form (1) belongs to the class $\mathcal{J}_{k,j}(q, \beta, A, B)$ if:

$$\sum_{m=2}^{\infty} \left\{ (\beta(B) + 1) \left( \theta_k^f[m] - \theta_q^f[m] \right) + \left| B \theta_k^f[m] - A \theta_q^f[m] \right| \right\} a_m \leq A - B.$$  \hfill (8)

where $-1 \leq B < A \leq 1, \beta \geq 0, k \in \mathbb{N}_0, j \in \mathbb{N}_0, k > j, q \in (0,1)$

**Proof.** It is sufficient to prove that

$$\frac{p(z) - 1}{A - B p(z)} < 1,$$

where

$$p(z) = \frac{R_k^f(z)}{R_q^f(z)} - \beta \frac{|R_k^f(z)|}{|R_q^f(z)|} - 1.$$

We obtain

$$\frac{p(z) - 1}{A - B p(z)} = \left| \frac{R_k^f(z) - R_q^f(z) - \beta e^{i\theta} |R_k^f(z)| - |R_q^f(z)|}{A R_q^f(z) - B |R_q^f(z)| - \beta e^{i\theta} |R_k^f(z)| - |R_q^f(z)|} \right|$$

$$= \left| \sum_{m=2}^{\infty} \left\{ \theta_k^f[m] - \theta_q^f[m] \right\} a_m z^m - \beta e^{i\theta} \sum_{m=2}^{\infty} \left\{ \theta_k^f[m] - \theta_q^f[m] \right\} a_m z^m \right|$$

$$= \frac{(A-B)z - \sum_{m=2}^{\infty} \left\{ \theta_k^f[m] - \theta_q^f[m] \right\} a_m |z|^m + \beta \sum_{m=2}^{\infty} \left\{ \theta_k^f[m] - \theta_q^f[m] \right\} a_m |z|^m}{(A-B)z - \sum_{m=2}^{\infty} \left\{ \theta_k^f[m] - \theta_q^f[m] \right\} a_m |z|^m + \beta |B| \sum_{m=2}^{\infty} \left\{ \theta_k^f[m] - \theta_q^f[m] \right\} a_m |z|^m} \cdot$$

This final statement is bounded above by one if

$$\sum_{m=2}^{\infty} \left\{ (\beta(B) + 1) \left( \theta_k^f[m] - \theta_q^f[m] \right) + \left| B \theta_k^f[m] - A \theta_q^f[m] \right| \right\} a_m \leq A - B.$$

hence, the proof is completed. \hfill $\square$

**Theorem 2.** Consider that $f \in \mathcal{T}$. Then, $f \in \mathcal{T} \mathcal{J}_{k,j}(q, \beta, A, B)$ if and only if:

$$\sum_{m=2}^{\infty} \left\{ (\beta(B) + 1) \left( \theta_k^f[m] - \theta_q^f[m] \right) + \left| B \theta_k^f[m] - A \theta_q^f[m] \right| \right\} a_m \leq A - B.$$

**Proof.** Since $\mathcal{T} \mathcal{J}_{k,j}(q, \beta, A, B) \subset \mathcal{J}_{k,j}(q, \beta, A, B)$ for functions $f \in \mathcal{T} \mathcal{J}_{k,j}(q, \beta, A, B)$ we can put: $\frac{p(z) - 1}{A - B p(z)} < 1$ where $p(z) = \frac{R_k^f(z)}{R_q^f(z)} - \beta \frac{|R_k^f(z)|}{|R_q^f(z)|} - 1$.

Then

$$\sum_{m=2}^{\infty} \left\{ \theta_k^f[m] - \theta_q^f[m] \right\} a_m z^m - \beta e^{i\theta} \sum_{m=2}^{\infty} \left\{ \theta_k^f[m] - \theta_q^f[m] \right\} a_m z^m \right\} \left\{ (A-B)z + \sum_{m=2}^{\infty} \left\{ B \theta_k^f[m] - A \theta_q^f[m] \right\} a_m z^m \right\}^{-1} \leq 1.$$
Since \( \text{Re}(z) \leq |z| \), then we get

\[
\text{Re} \left\{ \sum_{m=2}^{\infty} \left( \left( \theta_k^q[m] - \theta_q^f[m] \right) a_m z^m - \beta e^{i\theta} \left( \sum_{m=2}^{\infty} \left( \theta_k^q[m] - \theta_q^f[m] \right) a_m z^m \right) \right) \right\} < 1.
\]

Now taking \( z \) to be real and letting \( z \to 1^- \), we have

\[
\sum_{m=2}^{\infty} \left\{ (1 + \beta(1 - B)) \left( \theta_k^q[m] - \theta_q^f[m] \right) - \left| B \theta_k^q[m] - A \theta_q^f[m] \right| \right\} a_m \leq A - B.
\]

or equivalently

\[
\sum_{m=2}^{\infty} \left\{ (1 + \beta(1 + |B|)) \left( \theta_k^q[m] - \theta_q^f[m] \right) + \left| B \theta_k^q[m] - A \theta_q^f[m] \right| \right\} a_m \leq A - B.
\]

By this the proof is finished.

**Corollary 1.** A function \( f \in T \) is in the class \( T_{J,k,f}(q,\beta,A,B) \). Then:

\[
a_m \leq \frac{1}{\left\{ (1 + \beta(1 + |B|)) \left( \theta_k^q[m] - \theta_q^f[m] \right) + \left| B \theta_k^q[m] - A \theta_q^f[m] \right| \right\}}, \quad m \geq 2.
\]

The result of the function is sharp, as follows:

\[
f(z) = \frac{z}{\left\{ (1 + \beta(1 + |B|)) \left( \theta_k^q[m] - \theta_q^f[m] \right) + \left| B \theta_k^q[m] - A \theta_q^f[m] \right| \right\}}, \quad m \geq 2.
\]

That is, the function defined in (10) can achieve the equality.

**Distortion theorems**

**Theorem 3.** Consider the function \( f \) defined by (7) in the class \( T_{J,k,f}(q,\beta,A,B) \). Then:

\[
|f(z)| \geq |z| - \frac{A - B}{\left\{ (\beta(1 + |B|) + 1 \left( \theta_k^q[2] - \theta_q^f[2] \right) \right\}} |z|^2,
\]

and

\[
|f(z)| \leq |z| + \frac{A - B}{\left\{ (\beta(1 + |B|) + 1 \left( \theta_k^q[2] - \theta_q^f[2] \right) \right\}} |z|^2.
\]

The result is sharp.

**Proof.** From Theorem 2, let the function:

\[
\Omega(m) = \left\{ (\beta(1 + |B|) + 1 \left( \theta_k^q[m] - \theta_q^f[m] \right) \right\}
\]

Then, it is obvious that it is an increasing function of \( m(m \geq 2) \), therefore:

\[
\Omega(2) \sum_{m=2}^{\infty} |a_m| \leq \sum_{m=2}^{\infty} \Omega(m) |a_m| \leq A - B.
\]

That is

\[
\sum_{m=2}^{\infty} |a_m| \leq \frac{A - B}{\Omega(2)}.
\]

Thus, we get

\[
|f(z)| \leq |z| + |z|^2 \sum_{m=2}^{\infty} |a_m|,
\]

\[
|f(z)| \leq |z| + \frac{A - B}{\left\{ (\beta(1 + |B|) + 1 \left( \theta_k^q[2] - \theta_q^f[2] \right) \right\}} |z|^2.
\]
Likewise, we get
\[ |f(z)| \geq |z| - \sum_{m=2}^{\infty} |a_m| |z|^m \geq |f(z)| \geq |z| - |z|^2 \sum_{m=2}^{\infty} |a_m|, \]
\[ \geq |z| - \left\{ (\beta(1 + |B|) + 1 \left( \vartheta_q^k[2] - \vartheta_q^j[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^j[2]| \right\} |z|^2. \]

Lastly, we can achieve the equality for the function, as follows:
\[ f(z) = z - \left\{ (1 + \beta(1 + |B|) \left( \vartheta_q^k[2] - \vartheta_q^j[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^j[2]| \right\} z^2. \quad (11) \]

At \(|z| = r\) and \(z = re^{i(2k+1)\pi}(k \in \mathbb{Z})\). This ended the result.

**Theorem 4.** Let the function \(f\) be defined by (7) in the class \(\mathcal{T}_{\beta,k,j}(q, \beta, A, B)\). Then:
\[ |f'(z)| \geq 1 - \frac{2(A - B)}{(\beta(1 + |B|) + 1 \left( \vartheta_q^k[2] - \vartheta_q^j[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^j[2]|)} |z|, \]
and:
\[ |f'(z)| \leq 1 + \frac{2(A - B)}{(\beta(1 + |B|) + 1 \left( \vartheta_q^k[2] - \vartheta_q^j[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^j[2]|)} |z|. \]

The result is sharp.

**Proof.** Since \(\Omega(m)\) is an increasing function for \(m(m \geq 2)\), then from Theorem 2, we get
\[ \frac{\Omega(2)}{2} \sum_{m=2}^{\infty} |a_m| \leq \sum_{m=2}^{\infty} \frac{\Omega(m)}{m} |a_m| = \sum_{m=2}^{\infty} \Omega(m) |a_m| = \leq A - B, \]
that is:
\[ \sum_{m=2}^{\infty} |a_m| \leq \frac{2(A - B)}{\Omega(2)}. \]
Thus, we obtain
\[ |f'(z)| \leq 1 + |z| \sum_{m=2}^{\infty} m |a_m|, \]
\[ |f'(z)| \leq 1 + \frac{2(A - B)}{(\beta(1 + |B|) + 1 \left( \vartheta_q^k[2] - \vartheta_q^j[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^j[2]|)} |z|. \]
Likewise, we obtain:
\[ |f'(z)| \geq 1 - \sum_{m=2}^{\infty} m |a_m| \]
\[ \geq 1 - \frac{2(A - B)}{(\beta(1 + |B|) + 1 \left( \vartheta_q^k[2] - \vartheta_q^j[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^j[2]|)} |z|. \]
Lastly, we can notice that the affirmations of the theorem are sharp for the function \(f(z)\) defined by (11). This finishes the proof.

**Convexity, Radii of starlikeness and close-to-convexity**

**Theorem 5.** A function \(f\) of the from (7) belongs to the class \(\mathcal{T}_{\beta,k,j}(q, \beta, A, B)\). Then:
(i) \(f\) is starlike of order \(\zeta(0 \leq \zeta < 1)\) in \(|z| < r_1\) where:
\[ r_1 = \inf_{m \geq 2} \left\{ \left( (1+\beta(1+|B|) \left( \vartheta_q^k[m] - \vartheta_q^j[m] \right) + |B\vartheta_q^k[m] - A\vartheta_q^j[m]| \right) \right\} \times \left( \frac{1-\zeta}{m-\zeta} \right)^{\frac{1}{m-1}}. \quad (12) \]
(ii) \(f\) is convex of order \(\zeta(0 \leq \zeta < 1)\) in \(|z| < r_2\), where:
\[ r_2 = \inf_{m \geq 2} \left\{ \left( (1+\beta(1+|B|) \left( \vartheta_q^k[m] - \vartheta_q^j[m] \right) + |B\vartheta_q^k[m] - A\vartheta_q^j[m]| \right) \right\} \times \left( \frac{1-\zeta}{m(m-\zeta)} \right)^{\frac{1}{m-1}}. \quad (13) \]
(iii) $f$ is close to convex of order $0 \leq \zeta < 1$ in $|z| < r_3$, where:

$$r_3 = \inf_{m \geq 2}\left\{ \left(1+\beta(1+|B|) \left( \phi_q^k[m] - \phi_q^j[m] \right) + |B\phi_q^k[m] - A\phi_q^j[m]| \right) \right\} \frac{1}{(A-B)} \left( \frac{1-\zeta}{m-\zeta} \right)^{\frac{1}{m-1}}. $$

The function $f$ is provided by (10). All of these results are sharp.

**Proof.** We need to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \zeta \text{ where } |z| < r_1$$

where $r_1$ is specified by(12) . Indeed, we get from (7) that:

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \sum_{m=2}^{\infty} (m-1) \alpha_m |z|^{m-1}.$$ 

Hence, we obtain:

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \zeta,$$

if and only if:

$$\sum_{m=2}^{\infty} (m-\zeta) \alpha_m |z|^{m-1} \leq \frac{1}{(1-\zeta)}.$$ 

By Theorem 2, the relation (15) is true if:

$$\left( \frac{m-\zeta}{1-\zeta} \right) |z|^{m-1} \leq \left( \frac{1+\beta(1+|B|) \left( \phi_q^k[m] - \phi_q^j[m] \right) + |B\phi_q^k[m] - A\phi_q^j[m]|}{(A-B)} \right).$$

That is, if:

$$|z| \leq \left( \frac{(1+\beta(1+|B|) \left( \phi_q^k[m] - \phi_q^j[m] \right) + |B\phi_q^k[m] - A\phi_q^j[m]|}{(A-B)} \right) \frac{1}{m-\zeta},$$

for $m \geq 2$. Implies

$$r_1 = \inf_{m \geq 2}\left\{ \left(1+\beta(1+|B|) \left( \phi_q^k[m] - \phi_q^j[m] \right) + |B\phi_q^k[m] - A\phi_q^j[m]| \right) \right\} \frac{1}{(A-B)} \left( \frac{1-\zeta}{m-\zeta} \right)^{\frac{1}{m-1}}, m \geq 2.$$ 

This completes the proof (12).

For proving (13) and (14), we need only to show that:

$$1 + \frac{zf'(z)}{f(z)} - 1 \leq 1 - \zeta \quad (0 \leq \zeta < 1; |z| < r_2),$$

and

$$|f'(z) - 1| \leq 1 - \zeta \quad (0 \leq \zeta < 1; |z| < r_3),$$

respectively.

**Extreme points**

**Theorem 6.** Consider that $f_1(z) = z$, and:

$$f_m(z) = z - \left( \frac{A-B}{(\beta(1+|B|) \left( \phi_q^k[2] - \phi_q^j[2] \right) + |B\phi_q^k[2] - A\phi_q^j[2]|} \right)^{m-1}, m = 2, 3, . \text{ -}$$

Then, $\inf_{m \geq 2}(q, \beta, A, B)$ if and only if it can be written in the following form:

$$f(z) = \sum_{m=1}^{\infty} \omega_m f_m(z),$$

where

$$\omega_m \geq 0, \sum_{m=1}^{\infty} \omega_m = 1.$$

**Proof.** Assume that:

$$f(z) = \sum_{m=1}^{\infty} \omega_m f_m(z)$$

$$= z - \sum_{m=2}^{\infty} \omega_m \left( \frac{A-B}{(\beta(1+|B|) \left( \phi_q^k[2] - \phi_q^j[2] \right) + |B\phi_q^k[2] - A\phi_q^j[2]|} \right)^{m-1}. $$

2356
Then, by Theorem 2, we get:
\[
\sum_{m=2}^{\infty} \left\{ \frac{\left((\beta(1+|B|) + 1) \left( \vartheta_k^q[2] - \vartheta_j^q[2] \right) + \left| B \vartheta_k^q[2] - A \vartheta_j^q[2] \right| \right)}{\left(\beta(1+|B|) + 1\right) \left( \vartheta_k^q[2] - \vartheta_j^q[2] \right) + \left| B \vartheta_k^q[2] - A \vartheta_j^q[2] \right|} \right\} (A-B) \omega_m = (A-B) \sum_{m=2}^{\infty} \omega_m = (A-B)(1-\omega_1) \leq (A-B).
\]
Thus, in view of Theorem 2, we obtain \( f \in \mathcal{T} \mathcal{J}_{k,j}(q,\beta,A,B) \).

Contrariwise, let us assume that, \( f \in \mathcal{T} \mathcal{J}_{k,j}(q,\beta,A,B) \) then
\[
a_m \leq \frac{\left(\beta(1+|B|) + 1\right) \left( \vartheta_k^q[2] - \vartheta_j^q[2] \right) + \left| B \vartheta_k^q[2] - A \vartheta_j^q[2] \right|}{\left(\beta(1+|B|) + 1\right) \left( \vartheta_k^q[2] - \vartheta_j^q[2] \right) + \left| B \vartheta_k^q[2] - A \vartheta_j^q[2] \right|} a_m
\]
for
\[
\omega_1 = 1 - \sum_{m=2}^{\infty} \omega_m,
\]
we obtain
\[
f(z) = \sum_{m=1}^{\infty} \omega_m f_m(z).
\]
This completes the proof.

**Corollary 2.** The extreme points of the class \( \mathcal{T} \mathcal{J}_{k,j}(q,\beta,A,B) \) are given by
\[
f_1(z) = z,
\]
and
\[
f_m(z) = z - \frac{A-B}{\left(\beta(1+|B|) + 1\right) \left( \vartheta_k^q[m] - \vartheta_j^q[m] \right) + \left| B \vartheta_k^q[m] - A \vartheta_j^q[m] \right|} z^m, \quad m = 2,3,\ldots
\]

**Integral mean inequalities**

**Lemma 1.** [25] Let \( f \) and \( g \) be holomorphic functions in \( E^* \) with:
\[
f(z) < g(z),
\]
then for \( p > 0 \) and \( z = re^{i\theta}, (0 < r < 1), \)
\[
\int_0^{2\pi} |f(z)|^p d\theta \leq \int_0^{2\pi} |g(z)|^p d\theta. \tag{16}
\]

Now, we find the following result by taking Lemma 1.

**Theorem 7.** Suppose that \( f \in \mathcal{T} \mathcal{J}_{k,j}(q,\beta,A,B), p > 0, k > j, k \in \mathbb{N}, j \in \mathbb{N}_0, -1 \leq B < A \leq 1, \beta \geq 0, \) and \( f_2(z) \) is defined by
\[
f_2(z) = z - \frac{A-B}{\left(\beta(1+|B|) + 1\right) \left( \vartheta_k^q[2] - \vartheta_j^q[2] \right) + \left| B \vartheta_k^q[2] - A \vartheta_j^q[2] \right|} z^2,
\]
for \( z = re^{i\theta}, (0 < r < 1), \) we get
\[
\int_0^{2\pi} |f(z)|^p d\theta \leq \int_0^{2\pi} |f_2(z)|^p d\theta.
\]

**Proof.** For
\[
f(z) = z - \sum_{m=2}^{\infty} a_m z^m, \quad a_m \geq 0,
\]
the relation (16) is equivalent to prove that
\[
\int_0^{2\pi} \left| 1 - \sum_{m=2}^{\infty} a_m z^{m-1} \right|^p d\theta \leq \int_0^{2\pi} \left| 1 - \frac{A-B}{\left(\beta(1+|B|) + 1\right) \left( \vartheta_k^q[2] - \vartheta_j^q[2] \right) + \left| B \vartheta_k^q[2] - A \vartheta_j^q[2] \right|} z \right|^p d\theta.
\]
By using Lemma 1, it suffices to show that
\[ 1 - \sum_{m=2}^{\infty} a_m z^{m-1} < 1 - \frac{A-B}{(\beta(1+|B|) + 1) \left( \vartheta_q^k[2] - \vartheta_q^l[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^l[2]|} z. \]

By setting
\[ 1 - \sum_{m=2}^{\infty} a_m z^{m-1} = 1 - \frac{A-B}{(\beta(1+|B|) + 1) \left( \vartheta_q^k[2] - \vartheta_q^l[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^l[2]|} w(z). \]
and using (8), we get
\[
|w(z)| = \left| \sum_{m=2}^{\infty} \frac{\left( (\beta(1+|B|) + 1) \left( \vartheta_q^k[2] - \vartheta_q^l[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^l[2]| \right)}{A-B} a_m z^{m-1} \right|
\leq |z| \sum_{m=2}^{\infty} \frac{\left( (\beta(1+|B|) + 1) \left( \vartheta_q^k[2] - \vartheta_q^l[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^l[2]| \right)}{A-B} a_m
\leq |z| \sum_{m=2}^{\infty} \frac{\left( (\beta(1+|B|) + 1) \left( \vartheta_q^k[m] - \vartheta_q^l[m] \right) + |B\vartheta_q^k[m] - A\vartheta_q^l[m]| \right)}{A-B} a_m
\leq |z| < 1.

This completes the proof.

**Neighborhoods for the class \( \mathcal{T} \mathcal{J}_{k,j}(q, \beta, A, B) \)**

We define the \( \delta_1 \)-neighborhood of a function \( f \) in \( \mathcal{T} \) by
\[
N_{\delta_1}(f) = \left\{ g \in \mathcal{T} ; g(z) = z - \sum_{m=2}^{\infty} b_m z^m : \sum_{m=2}^{\infty} m |a_m - b_m| \leq \delta_1 \right\}. \tag{16}
\]

In particular, for \( e(z) = z \),
\[
N_{\delta_1}(e) = \left\{ g \in \mathcal{T} ; g(z) = z - \sum_{m=2}^{\infty} b_m z^m \text{ and } \sum_{m=2}^{\infty} m |b_m| \leq \delta_1 \right\}. \tag{17}
\]

On the other hand, a function \( f(z) \) defined by (7) is said to be in the class \( \mathcal{T} \mathcal{J}_{k,j}(q, \beta, A, B) \) if there exists a function \( g \in \mathcal{T} \mathcal{J}_{k,j}(q, \beta, A, B) \) such that
\[
\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - Y, \quad 0 \leq Y < 1 \tag{18}
\]

**Theorem 8.** If
\[
\left\{ (\beta(1+|B|) + 1) \left( \vartheta_q^k[m] - \vartheta_q^l[m] \right) + |B\vartheta_q^k[m] - A\vartheta_q^l[m]| \right\}
\geq \left\{ (\beta(1+|B|) + 1) \left( \vartheta_q^k[2] - \vartheta_q^l[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^l[2]| \right\}, \quad m \geq 2.
\]
and
\[
\delta_1 = \frac{2(A-B)}{\left( (\beta(1+|B|) + 1) \left( \vartheta_q^k[2] - \vartheta_q^l[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^l[2]| \right)},
\]
then
\[ \mathcal{T} \mathcal{J}_{k,j}(q, \beta, A, B) \subset N_{\delta_1}(e) \]

**Proof.** Let \( f \in \mathcal{T} \mathcal{J}_{k,j}(q, \beta, A, B) \). Then from Theorem 2 and the condition
\[
\left\{ (\beta(1+|B|) + 1) \left( \vartheta_q^k[m] - \vartheta_q^l[m] \right) + |B\vartheta_q^k[m] - A\vartheta_q^l[m]| \right\}
\geq \left\{ (\beta(1+|B|) + 1) \left( \vartheta_q^k[2] - \vartheta_q^l[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^l[2]| \right\}, \quad m \geq 2.
\]
We get,
\[
\left\{ (\beta(1+|B|) + 1) \left( \vartheta_q^k[2] - \vartheta_q^l[2] \right) + |B\vartheta_q^k[2] - A\vartheta_q^l[2]| \right\} \sum_{m=2}^{\infty} a_m \leq
\]
\[
\left\{ (\beta(1 + |B|) + 1) \left( \theta^k_q[m] - \theta^l_q[m] \right) + \left| B\theta^k_q[2] - A\theta^l_q[2] \right| \right\} \sum_{m=2}^{\infty} a_m \leq (A - B),
\]
which implies
\[
\sum_{m=2}^{\infty} a_m \leq \frac{(A - B)}{\left\{ (\beta(1 + |B|) + 1) \left( \theta^k_q[2] - \theta^l_q[2] \right) + \left| B\theta^k_q[2] - A\theta^l_q[2] \right| \right\}}.
\]
(19)

By using Theorem 2 with (19), we obtain
\[
\left\{ (\beta(1 + |B|) + 1) \left( \theta^k_q[2] - \theta^l_q[2] \right) + \left| B\theta^k_q[2] - A\theta^l_q[2] \right| \right\} \sum_{m=2}^{\infty} a_m \leq (A - B)
\]
\[
2 \left\{ (\beta(1 + |B|) + 1) \left( \theta^k_q[2] - \theta^l_q[2] \right) + \left| B\theta^k_q[2] - A\theta^l_q[2] \right| \right\} \sum_{m=2}^{\infty} a_m \leq 2(A - B)
\]
\[
\sum_{m=2}^{\infty} m a_m \leq \frac{2(A - B)}{\left\{ (\beta(1 + |B|) + 1) \left( \theta^k_q[2] - \theta^l_q[2] \right) + \left| B\theta^k_q[2] - A\theta^l_q[2] \right| \right\} - (A - B)}.
\]

By (16) we get \( f \in N_{\delta_1}(e) \).

This completes the proof of Theorem 8.

**Theorem 9.** If \( g \in T_{J,k,j}(q, \beta, A, B) \) and
\[
Y = 1 - \frac{\delta_1}{2} \frac{\left\{ (\beta(1 + |B|) + 1) \left( \theta^k_q[2] - \theta^l_q[2] \right) + \left| B\theta^k_q[2] - A\theta^l_q[2] \right| \right\}}{(\beta(1 + |B|) + 1) \left( \theta^k_q[2] - \theta^l_q[2] \right) + \left| B\theta^k_q[2] - A\theta^l_q[2] \right|}.
\]
(20)

Then \( N_{\delta_1}(e) \subset T_{J,k,j}(q, \beta, A, B) \).

**Proof.** Let \( f \) be in \( N_{\delta_1}(e) \). We find by (16) that
\[
\sum_{m=2}^{\infty} m |a_m - b_m| \leq \delta_1,
\]
which means the coefficient of inequality
\[
\sum_{m=2}^{\infty} |a_m - b_m| \leq \frac{\delta_1}{2}.
\]
(21)

It follows that, since \( g \in T_{J,k,j}(q, \beta, A, B) \), then from (19) we obtain
\[
\sum_{m=2}^{\infty} b_m \leq \frac{(A - B)}{\left\{ (\beta(1 + |B|) + 1) \left( \theta^k_q[2] - \theta^l_q[2] \right) + \left| B\theta^k_q[2] - A\theta^l_q[2] \right| \right\}}.
\]
(22)

Using (21) and (22), we get
\[
\frac{|f(z)|}{|g(z)|} - 1 \leq \frac{\sum_{m=2}^{\infty} |a_m - b_m|}{1 - \sum_{m=2}^{\infty} b_m} \leq \frac{\delta_1}{2} \frac{(A - B)}{\left\{ (\beta(1 + |B|) + 1) \left( \theta^k_q[2] - \theta^l_q[2] \right) + \left| B\theta^k_q[2] - A\theta^l_q[2] \right| \right\}}.
\]
\[
\leq \frac{\delta_1}{2} \frac{\left\{ (\beta(1 + |B|) + 1) \left( \theta^k_q[2] - \theta^l_q[2] \right) + \left| B\theta^k_q[2] - A\theta^l_q[2] \right| \right\}}{(\beta(1 + |B|) + 1) \left( \theta^k_q[2] - \theta^l_q[2] \right) + \left| B\theta^k_q[2] - A\theta^l_q[2] \right|} = 1 - Y.
\]
provided that \( Y \) is given by (20), hence, by condition (18), \( f \) in \( \mathcal{J}_{k,j}^q(\beta, A, B) \) is given by (9).

References