On Soft bc-Open Sets in Soft Topological Spaces

Saif Z. Hameed*, Adiya K. Hussein
Department of Mathematics, College of Basic Education Mustansiriyah University, Baghdad, Iraq

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Abstract
In this paper, we offer and study a novel type of generalized soft-open sets in soft topological spaces, named soft bc-open sets. Relationships of this set with other types of generalized soft-open sets are discussed, definitions of soft bc-neighborhood, soft bc-closure and soft bc-interior are introduced, and its properties are investigated. Also, we introduce and explore several characterizations and properties of this type of sets.

Keywords: soft b-open set, soft bc-open set, soft bc-nbd, soft bc-interior, soft bc-closure.

1. Introduction and Preliminaries
The concept of soft set theory was instigated and applied by Molodtsosv [1,2] as a mathematical device for dealing with uncertainties. In a previous work [3], Shaber and Naz defined soft topological spaces and soft open sets. Soft β-open sets were introduced and studied by several authors, including the soft α-open [4], soft preopen [5], soft semi open [4], and soft regular open sets [6]. In another study [7], Akdag and Ozkan realized the soft b-open sets and soft continuity. The concept of bc-open sets was introduced by Ibrahim [8].

Let (Z, τ, A) be a soft topological space, where A is any set of parameters. The soft closure (resp. soft interior) [9] of a soft set (P, A) is denoted by cl(P, A) (resp. int(P, A)). A subset (P, A) is said to be a β-open [4](resp. soft α-open[5], soft preopen[4], soft semi-open[6] and soft regular) set, if: (P, A) ⊆ cl(int(cl((P, A)))) (resp. (P, A) ⊆ int(cl(int((P, A))))) (P, A) ⊆ int(cl((P, A))), (P, A) ⊆ cl(int((P, A))) and (P, A) = int (cl((P, A))). We denote the family of all soft sets over X by SS(Z, A).

Definition 1.1[3]. Let τ be a collection of soft open sets over Z, then τ is said to be soft topological space if (1) ∅ and X belong to τ, (2) The union of any subcollection

*Email: saif.zuhar.edbs@uomustansiriyah.edu.iq
of soft sets of \( \tau \) belongs to \( \tau \), and (3) the intersection of any two soft sets in \( \tau \) belongs to \( \tau \). We named the triple \((Z, \tau, A)\) by \(STS\).

**Definition 1.2** [9]. The soft set \((P, A) \in SS(Z, A)\) is called a soft point in \(Z\), denoted by \(e_{L}\), if for the element \(e \in A\), \(F(e) \neq \emptyset \) and \(F(e') = \emptyset\) for all \(e' \in A \backslash \{e\}\). The set of all soft points of \(Z\) is denoted by \(SP(Z)\).

**Definition 1.3** [7]. If \((P, A) \in SS(Z, A)\) then it is called

1. Soft \(b - \) open set (briefly \(sb - \) open set), iff \((P, A) \subset int(cl((P, A))) \cup cl(int((P, A)))\).
2. Soft \(b - \) closed (briefly \(sb - \) closed) set, iff \((P, A) \supset int(cl((P, A))) \cap cl(int((P, A)))\).

**Definition 1.4** [6, 7]. Let \((P, A)\) be a soft set of a \(STS\) \((Z, \tau, A)\), then

1. Soft semi-interior of a soft set \(P, A\) in \(Z\) is denoted by \(sSint((P, A)) = \bigcup\{(W, A) : (W, A)\) is a soft semi-open set and \((W, A) \subset (P, A)\}\}.
2. Soft semi-closure of a soft set \(P, A\) in \(Z\) is denoted by \(sScl((P, A)) = \bigcap\{(L, A) : (L, A)\) is a soft semi-closed set and \((P, A) \subset (L, A)\}\}.
3. Soft \(b - \) interior of a soft set \(P, A\) in \(Z\) is denoted by \(sSint((P, A)) = \bigcup\{(W, A) : (W, A)\) is a soft \(b\) - open set and \((W, A) \subset (P, A)\}\}.
4. Soft \(b - \) closure of a soft set \(P, A\) in \(Z\) is denoted by \(sScl((P, A)) = \bigcap\{(L, A) : (L, A)\) is a soft \(b\) - closed set and \((P, A) \subset (L, A)\}\}.

Clearly, \(sScl(P, A)\) (resp. \(sScl\) \((P, A)\)) is the smallest soft \(b\) - closed (resp. soft semi-closed) set over \(Z\) which contains \((P, A)\), and \(sSint\) \((P, A)\) (resp. \(sSint\) \((P, A)\)) is the largest soft \(b\) - open (resp. semi-open) set over \(Z\) which is contained in \((P, A)\).

We will denote the family of all soft \(b\) - open (resp., soft semi-open) sets and soft \(b\) - closed ((resp., soft semi-closed) sets of a soft topological space by \(SO(Z)\) (resp., \(SSO(Z)\) \(SbC(Z)\) (resp., \(SSC(Z)\)).

**Definition 1.5** [10]. Let \((Z, \tau, A)\) be a \(STS\), and \(x, y \in Z\), such that \(x \neq y\). If there exist soft open sets \((P, A)\) and \((S, A)\), such that \(x \in (P, A), y \in (S, A)\), then \((Z, \tau, A)\) is called a soft \(T_1\) - space.

**Theorem 1.6** [10]. Let \((Z, \tau, A)\) be \(STS\). Then each soft point is a soft closed if and only if \((Z, \tau, A)\) is a soft \(T_1\) - space.

**Definition 1.7** [11]. An \(STS\) \((Z, \tau, A)\) is called a soft locally indiscrete, if every soft open set over \(Z\) is a soft closed set over \(Z\).

2. Soft \(bc - \) open sets

Now, we give a new family of soft \(b - \) open sets named soft \(bc - \) open sets in an \(STS\) and study some of its basic properties.

**Definition 2.1** A subset \((P, A)\) of \(STS\) \((Z, \tau, E)\) is named soft \(bc - \) open (\(sbc - \) open) if, for any \(x \in (P, A) \in SO(Z)\), there is a soft closed set \((S, A)\), such that \(x \in (S, A) \subset (P, A)\). The complement of \(P, A\) is named soft \(bc - \) closed (\(sbc - \) closed).

The collection of all soft \(bc - \) open sets in \(Z\) is denoted by \(SbcO(Z)\) and the collection of all soft \(bc - \) closed sets in \(Z\) is named \(SbcC(Z)\).

**Theorem 2.2** A soft subset \((P, A)\) of \(STS\) \((Z, \tau, A)\) is soft \(bc - \) open iff \((P, A)\) is \(sb - \) open and it is a union of soft closed sets.

**Proof.** (\(\Rightarrow\)) Let \((P, A)\) be a soft \(bc - \) open set. Then \((P, A)\) is \(sb - \) open set and for each \(x \in (P, A)\) there is a soft closed set \((L, A)\), such that \(x \in (L, A) \subset (P, A)\). Then we get \(\bigcup\{x \in (P, A) = (P, A) \subset (L, A) \subset (P, A)\). Thus, \((P, A) = \bigcup(L, A)\), where \((P, A)\) is a soft closed set for each \(x \in (P, A)\).

(\(\Leftarrow\)) Direct form the definition of soft \(bc - \) open.

**Corollary 2.3** For a \(STS\) \((Z, \tau, A)\), if \((L, A)\) is \(sb - \) open set over \(X\), then \((L, A)\) is an \(sbc - \) open if \((L, A)\) is a soft closed set.

**Proposition 2.4** A soft subset \((H, A)\) of an \(STS\) \((Z, \tau, A)\) is \(sbc - \) closed if and only if \((H, A)\) is a soft \(b - \) closed set and it is an intersection of soft open sets.

**Proof.** It is obvious.

**Remark 2.5** Every \(sbc - \) open set of a space \(Z\) is soft \(b - \) open, but the converse is not true in general, as shown by the following example.

**Example 2.6** Let \(Z = \{v_1, v_2, v_3\}\), \(A = \{e_1, e_2, e_3\}\) and \(\tau = \{\emptyset, Z, (P_1, A), (P_2, A), (P_3, A), (P_4, A)\}\) where \((P_1, A), (P_2, A), (P_3, A), (P_4, A)\) are soft sets over \(Z\), defined as follows:
\[(P_1, A) = \{(e_1, \overline{Z}), (e_2, \{v_2, v_3\}), (e_3, \{v_1, v_2\})\},\]
\[(P_2, A) = \{(e_1, \overline{\emptyset}), (e_2, \{v_1\}), (e_3, \{v_3\})\},\]
\[(P_3, A) = \{(e_1, \{v_2\}), (e_2, \{v_1, v_3\}), (e_3, \{v_1\})\},\]
\[(P_4, A) = \{(e_1, \{v_2\}), (e_2, \{v_3\}), (e_3, \emptyset)\}.
\]

Then, \(\tau\) defines a soft topology on \(Z\).

The soft closed sets are \(\overline{\emptyset}, (P_1, A)^c, (P_2, A)^c, (P_3, A)^c, (P_4, A)^c\),
where \((P_1, A)^c = \{(e_1, \emptyset), (e_2, \{v_1\}), (e_3, \{v_3\})\} = (P_2, A)\)
\((P_2, A)^c = \{(e_1, \overline{Z}), (e_2, \{v_2, v_3\}), (e_3, \{v_1, v_2\})\} = (P_1, A)\)
\((P_3, A)^c = \{(e_1, \{v_1, v_3\}), (e_2, \{v_2\}), (e_3, \{v_1, v_2\})\},\)
\((P_4, A)^c = \{(e_1, \{v_1, v_3\}), (e_2, \{v_2, v_3\}), (e_3, \overline{Z})\}.
\]

The family of \(SbOS(Z) = \{(P_1, A), (P_2, A), (P_3, A), (P_4, A)\}.

The family of \(SbcOS(Z) = \{(P_1, A), (P_2, A), (P_3, A)\}.

Then \((P_4, A) \in SbOS(Z)\), but \((P_4, A) \not\in SbcOS(Z)\).

By Remark 2.8 and the above Remark 2.5, we have the following implications:

\[
\text{Soft regular set} \Rightarrow \text{Soft open set} \Rightarrow \text{Soft semi-open set} \Rightarrow \text{Soft \(\alpha\)-open set} \Rightarrow \text{Soft \(\beta\)-open set}
\]

\[
\text{Soft pre-open set} \Rightarrow \text{Soft \(b\)-open set} \Rightarrow \text{Soft \(\beta\)-open set} \Rightarrow \text{Soft \(\alpha\)-open set} \Rightarrow \text{Soft \(b\)-open set}
\]

\[\text{Proposition 2.7} \text{ An arbitrary union of } \text{sbc - open sets is } \text{sbc - open set.}\]

\[\text{Proof.} \text{ Suppose that } \{(P, A)_\lambda : \lambda \in \Delta\} \text{ is a family of soft } bc - \text{open sets in } (Z, \tau, A) \text{. Then } (P, A)_\lambda \text{ is soft } \text{bc - open set for each } \lambda \in \Delta \text{. So, } \bigcup (P, A)_\lambda \text{ is soft } \text{bc - open set. Let } x \in \bigcup \{(P, A)_\lambda : \lambda \in \Delta\} \text{, so } x \in (L, A)_\lambda \text{ for some } \lambda \in \Delta \text{. Since } (P, A)_\lambda \text{ is soft } bc - \text{open for each } \lambda \text{, then there is a soft closed set } (L, A) \text{ such that } \]
\[x \in (L, A) \subset (P, A)_\lambda \subset \bigcup \{(P, A)_\lambda : \lambda \in \Delta\}, \text{ so } x \in (L, A) \subset \bigcup \{(P, A)_\lambda : \lambda \in \Delta\}. \text{ Therefore, } \bigcup \{(P, A)_\lambda : \lambda \in \Delta\} \text{ is soft } bc - \text{open set.}\]

Now we show that the intersection of two \(bc\)-open sets is not necessarily \(bc\)-open.

\[\text{Example 2.8.} \text{ Let the } STS(Z, \tau, A) \text{ as in Example 2.6, then } (P_1, A) \in SbOS(Z) \text{ and } (P_3, A) \in SbOS(Z) \text{, but } (P_1, A) \cap (P_3, A) = (P_4, A) \notin SbcO(Z).
\]

\[\text{Remark 2.9.} \text{ From the above example we notice that the family of all } bc - \text{open subset of a space } Z \text{ is a supra topology and thus it is not a topology in general.}\]

The following result gives a condition under which the family of all \(bc - open sets became a topology on } Z.\]

\[\text{Proposition 2.10.} \text{ If the collection } SbOS(Z) \text{ is a topology on } Z, \text{ then } SbOS(Z) \text{ is also a topology on } Z.
\]

\[\text{Proof.} \text{ It is clear that } \overline{\emptyset}, \overline{Z} \in SbOS(Z) \text{ and, by Proposition 2.7, the union of any subset of } SbOS(Z) \text{ is } \text{sbc - open. Now, let } (P, A) \text{ and } (S, A) \text{ be two } \text{sbc - open sets, then } (P, A) \text{ and } (P, A) \text{ are soft } bc - \text{open sets. Since } SbOS(Z) \text{ is a topology on } Z, \text{ so } (P, A) \cap (S, A) \text{ is soft } bc - \text{open. If } x \in (S, A) \cap (P, A) \text{, then } x \in (P, A) \text{ and } x \in (S, A). \text{ So there exist two soft closed sets } (L, A) \text{ and } (K, A), \text{ such that } x \in (L, A) \subset (S, A) \text{ and } x \in (K, A) \subset (P, A). \text{ This implies that } x \in (L, A) \cap (K, A) \subset (S, A) \cap (P, A). \text{ Since any intersection of soft closed sets is soft closed, then } (L, A) \cap (K, A) \text{ is a closed set. Thus, } (P, A) \cap (S, A) \text{ is } \text{sbc - open set.}\]

\[\text{Theorem 2.11} \text{ A soft set } (P, A) \text{ of a } STS(Z, \tau, A) \text{ is a soft } bc - \text{open set iff, for each } x \in (P, A), \text{ there is a soft } bc - \text{open set } (S, A) \text{ such that } x \in (S, A) \subset (P, A).\]

\[\text{Proof.} \text{ Suppose that } (P, A) \text{ is soft } bc - \text{open in the space } Z, \text{ then for each } x \in (P, A), \text{ put } (P, A) = (S, A) \text{ is } \text{sbc - open set containing } x \text{ such that } x \in (P, A) \subseteq (S, A). \text{ Conversely, assume that for any } x \in (P, A), \text{ there is a } \text{sbc - open set } (S, A) \text{ such that } x \in (S, A) \subseteq (P, A). \text{ Thus, } (P, A) = U(S, A)_x \text{ where } (S, A)_x \in SbOS(Z) \text{ for each } x. \text{ Hence, } (P, A) \text{ is } \text{sbc - open set.}\]
Theorem 2.12 Let $(Z, \tau, A)$ be soft $T_1$-space, then $(P, A)$ is $sbc$-open set iff $(P, A)$ is a soft $bc$-open set.

Proof Suppose that $(Z, \tau, A)$ is soft $T_1$-space and $(P, A)$ is $sbc$-open set. If $(P, A) = \emptyset$, then $(P, A) \in Sbc(Z)$. If $(P, A) \neq \emptyset$, let $x \in (P, A)$. Since $(Z, \tau, A)$ is soft $T_1$-space, then by Theorem 1.6, each soft point is a closed set and, hence, $x \in \{x\} \subset (P, A)$. Therefore, $(P, A)$ is an $sbc$-open, thus $SbO(Z) \subset SbcO(Z)$. But $SbcO(Z) \subset SbO(Z)$ Hence, $SbO(Z) = SbcO(Z)$.

Proposition 2.13 If $(Z, \tau, A)$ is soft locally indiscrete, then $SSO(Z) \subset SbcO(Z)$.

Proof Let $(P, A)$ be any soft subset of $STS(Z, \tau, A)$ and $(P, A) \in SSO(Z)$, if $(P, A) = \emptyset$, then $(P, A) \in SbcO(Z)$. If $(P, A) \neq \emptyset$, then $(P, A) \subset cl(int((P, A)))$. Since $(Z, \tau, A)$ is soft locally indiscrete, then int$(P, A)$ is soft closed, so $int(P, A) \subset (P, A)$. This implies that for each $x \in (P, A)$, $x \in (int(P, A)) \subset (P, A)$. Therefore, $(P, A)$ is $sbc$-open set. Hence $SSO(Z) \subset SbcO(Z)$.

Theorem 2.14 Let $\{(P, A)_\alpha : \alpha \in \Delta\}$ be a collection of sets such that $sbc$-closed sets in a soft topological space $(Z, \tau, A)$. Then $\bigcap\{(P, A)_\alpha : \alpha \in \Delta\}$ is soft $bc$-closed.

Proof The proof follows from Proposition 2.7.

3. Some Properties of Soft $bc$ - Open Sets

In this section, we provide some soft topological operations on soft sets and discuss its properties.

Definition 3.1 Let $(Z, \tau, A)$ be an $STS$ and $x \in \bar{Z}$. Then, a soft set $(P, A)$ is said to be soft $bc$-neighborhood (briefly, soft $bc$-nbhd) of $x$, if there exists a soft $bc$-open set $(K, A)$ such that $x \in (K, A) \subset (P, A)$.

Proposition 3.2 For an $STS(Z, \tau, A)$, a soft set $(P, A)$ is $sbc$-open iff it is a soft $bc$-neighborhood of each of its points.

Proof Let $(P, A) \subset \bar{Z}$ be a soft $sbc$-open set, since for every $x \in (P, A), x \in (P, A) \subset (P, A)$ and $(P, A)$ is $sbc$-open, this shows that $(P, A)$ is a soft $bc$-neighborhood of each of its points.

Conversely, suppose that $(P, A)$ is a soft $bc$-neighborhood of each of its points. Then for each $x \in (P, A)$, there exists $(S, A)_x \in SbcO(Z)$ such that $(S, A)_x \subset (P, A)$. Then $(P, A) = \bigcup\{(S, A)_x : x \in (P, A)\}$. Since each $(S, A)_x$ is $sbc$-open. It follows that $(P, A)$ is $sbc$-open.

Proposition 3.3 Every soft $bc$-neighborhood of a point is soft $b$-neighborhood.

Proof It is obvious from the fact that every $sbc$-open set is $sb$-open.

Definition 3.4 Let $(P, A)$ be soft set of a $STS(Z, \tau, A)$, then a point $x \in Z$ is called soft $bc$-interior point of $(P, A)$, if there exists an $sbc$-open set $(U, A)$ such that $x \in (U, A) \subset (P, A)$. The set of all soft $bc$-interior points of $(P, A)$ is denoted by $sbcInt(P, A)$.

Proposition 3.5 Let $(P, A)$ be a soft set of $Z$, then $sbcInt(P, A) \subset sbInt(P, A)$.

Proof Since $sbc$-open set is $sb$-open, so the proof holds.

Definition 3.6 Let $(P, A)$ be soft set of a $STS(Z, \tau, A)$, then the soft $bc$-closure of $(P, A)$, denoted by $sbcCl(P, A)$, is the intersection of all $sbc$-closed sets containing $(P, A)$.

In the following theorem we provide some properties of $sbc$-interior of a soft set.

Theorem 3.7 Let $(Z, \tau, A)$ be $STS$ and let $(P, A)$ and $(M, A)$ be soft sets over $Z$. Then

1) $sbcInt(P, A)$ is the union of all $sbc$-open sets which are contained in $(P, A)$.
2) $sbcInt(P, A)$ is $sbc$-open set in $Z$.
3) $(P, A)$ is $sbc$-open iff $P, A) = sbcInt(P, A)$.
4) $sbcInt(sbcInt(P, A)) = sbcInt(P, A)$.
5) $sbcInt(\emptyset) = \emptyset$ and $sbcInt(Z) = Z$.
6) $sbcInt(P, A) \subset (P, A)$.
7) If $(P, A) \subset (M, A)$, then $sbcInt(P, A) \subset sbcInt(M, A)$.
8) If $(P, A) \cap (M, A) = \emptyset$, then $sbcInt(P, A) \subset sbcInt(M, A)$.
9) $sbcInt(P, A) \cup sbcInt(M, A) \subset sbcInt((P, A) \cup (M, A))$.
10) $sbcInt((P, A) \cap (M, A)) \subset sbcInt(P, A) \cap sbcInt(M, A)$.

Proof The proofs of these facts are easy, so we will only prove the point number 7:

Suppose that $x \in Z$, $x \in sbcInt(P, A)$, then by Definition 3.4, there is a set $(U, A)$ such that $x \in (U, A) \subset (P, A) \subset (P, A)$, thus $x \in sbcInt(P, A)$.
Theorem 3.8 Let \((P, A)\) be any soft set in a STS \((Z, \tau, A)\), then \(x \in sbcCl(Z)\) if and only if 
\((P, A) \cap (U, A) = \emptyset\) for any \(sbc - \) open set \((U, A)\) containing \(x\).

Proof. (\(\Rightarrow\)) Let \(x \in sbcCl(P, A)\). Suppose that 
\((P, A) \cap (U, A)_x = \emptyset\), where \((U, A)_x \in sbcO(Z)\) containing \(x\). Hence, 
\((P, A) \subseteq (U, A)_x\), where \((U, A)_x \in sbcC(P, A)\). Hence, 
\(x \notin sbcCl(P, A)\), which is a contradiction.

(\(\Leftarrow\)) Let \(x \notin sbcCl(P, A)\), then \(x \notin (U, A)_x\), where \((L, A) \in sbcC(Z)\) and \((P, A) \subseteq (L, A)\) for each \((L, A)\). Hence, \(x \notin (\cap (L, A))_x^c\), where \((\cap (L, A))_x^c \in sbcO(Z)\) containing \(x\). Now, we have 
\((P, A) \cap (\cap (L, A))_x^c \subseteq (\cap (L, A))_x^c \cap (\cap (L, A))_x^c = \emptyset\).

Theorem 3.9 Let \((P, A)\) be any soft subset of an STS \((Z, \tau, A)\) if \((P, A) \cap (U, A) \neq \emptyset\) for any soft closed set \((L, A)\) containing \(x\), then \(x \in sbcCl(P, A)\).

Proof. Suppose that \((U, A)_x \in sbcO(Z)\) containing \(x\), then by definition (2.1), there is \((L, A)\) soft closed set such that \(x \in (L, A) \subseteq (U, A)\). So, by the hypothesis, \((P, A) \cap (L, A) \neq \emptyset\). Hence, 
\((P, A) \cap (U, A) \neq \emptyset\) for any \(sbc - \) open set \((U, A)_x\). Therefore, \(x \in sbcCl(P, A)\).

Theorem 3.10. Let \((Z, \tau, A)\) be an STS and let \((P, A)\) and \((M, A)\) be soft sets over \(Z\). Then

1) \(sbcCl(P, A)\) is the intersection of all \(sbc - \) closed sets which are containing \((P, A)\).

2) \((P, A) \subseteq sbcCl(P, A)\).

3) \(sbcCl(P, A)\) is \(sbc - \) closed set in \(Z\).

4) \((P, A)\) is \(sbc - \) closed iff \((P, A) = sbcCl(P, A)\).

5) \(sbcCl(sbcCl(P, A)) = sbcCl(P, A)\).

6) \(sbcCl(\emptyset) = \emptyset\) and \(sbcCl(Z) = Z\).

7) If \((P, A) \subseteq (M, A)\), then \(sbcCl(P, A) \subseteq sbcCl(M, A)\).

8) If \(sbcCl(P, A) \cap sbcCl(M, A) = \emptyset\), then \((P, A) \cap (M, A) = \emptyset\).

9) \(sbcCl(P, A) \cup sbcCl(M, A) \subseteq sbcCl((P, A) \cup (M, A))\).

10) \(sbcCl((P, A) \cap (M, A)) \subseteq sbcCl(P, A) \cap sbcCl(M, A)\).

Proof. It is obvious.

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References


