Numerical Solutions for the Optimal Control Governing by Variable Coefficients Nonlinear Hyperbolic Boundary Value Problem Using the Gradient Projection, Gradient and Frank Wolfe Methods

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Abstract
This paper is concerned with studying the numerical solution for the discrete classical optimal control problem (NSDCOCP) governed by a variable coefficients nonlinear hyperbolic boundary value problem (VCNLHBVP). The NSDCOCP is solved by using the Galerkin finite element method (GFEM) for the space variable and implicit finite difference scheme (GFEM-IFDS) for the time variable to get the NS for the discrete weak form (DWF) and for the discrete adjoint weak form (DSAWF) While, the gradient projection method (GRPM), also called the gradient method (GRM), or the Frank Wolfe method (FRM) are used to minimize the discrete cost function (DCF) to find the NSOC. Within these three methods, the Armijo step option (ARMSO) or the optimal step option (OPSO) are used to improve the discrete classical control (DSCC). Finally, some illustrative examples for the problem are given to show the accuracy and efficiency of the methods.

Keywords: Numerical Classical Optimal Control, Galerkin Finite Element Method, Gradient Method, Gradient Projection Method, Frank Wolfe Method.

الحلول العددية لسيطرة تمثيلية تتحكمها مسألة قيم حدودية من نوع القطع الزائد غير خطية ذات معاملات متغيرة باستخدام طريقة مسقط الانحدار ، طريقة الانحدار وطريقة فرانك فولف.

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الخلاصه
ينتناول هذا البحث دراسة الحل العددي لمسألة سيطرة تمثيلية من النوع المحدد بحكمها مسألة قيم حدودية زائدة غير خطية ذات معاملات متغيرة. تم إيجاد الحل لمسألة السيطرة المشتملة من النقط المدختة بواسطة مجزء تقريب كاليريكن للمنحنيات المتوسطة للتوترات المهينة والتوترات المتغيرة للمستويات الغير صريحة لمتغير الزمن لإيجاد الحل لمعادلة حالة المضاعفة للصيغة المتماثلة من النقط المدختة الهيكل المرافق من النقط المدختة للمصغرة المتماثلة للصيغة المصغرة للحالة ككل مع كل من طريقة فرانك فولف أو الانحدار أو طريقة مسقط الانحدار لإيجاد سيطرة تمثيلية تقريبية من النقط المدختة من خلال إيجاد قيمة غير لامعة لدالة الهيكل المذكورة، ضمن هذه الطرق الثلاث. تم استخدام خطوة انرخيو أو الخطوة المثلى لتحسين قيمة

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1. Introduction

Optimization problems have wide applications in medicine, sciences and many other fields [1, 2]. These applications are usually governed by partial differential equations (PDEs) or ordinary differential equations (ODEs).

Many researchers investigated the numerical solution of optimal control problems (NSOCPs) governed by nonlinear elliptic PDEs [3], semilinear parabolic PDEs [4], one dimensional linear hyperbolic PDEs with constant coefficients (LHPDES) [5], two dimensional linear and nonlinear hyperbolic PDEs with constant coefficients [6-9], two dimensional linear hyperbolic PDEs but with variable coefficients [10], or by one dimensional nonlinear ODEs [11]. The outcomes of these works have driven us to focus our interest on investigating the NSDCOC governed by the VCNLHBVP.

In this paper, the continuous classical optimal control problem (COCOCP) described by the VCNLHBVP is discretized by applying the Galerkin finite element method (GFEM) for the space variable and implicit finite difference scheme (GFEM-IFDS) for the time variable to get the DSCOCP ((the discrete weak form (DWF) for the VCNLHBVP and the discrete cost function (DFC)). To find such solutions, we should discuss the existence and the uniqueness theorem for the NS for the DWF. The proof of the existence theorem for the discrete classical optimal control (DCOC) and the necessary conditions of the problem are studied in a previous article [9] and they are all needed here. On the other hand, the DSCOCP is found numerically by using the GFEM-IFDS to find the NS of the DWF and then the DCOC by solving the optimization problem (the minimum of DCF) by using, separately, each one of the optimization methods; the GM, the GPM and the FWM. Within these three methods, the ARSO or the OPSO are used, separately, to get better direction of the optimal search. Some illustrative examples for this problem are given to show the performance of each of these methods.

2. Statement of the COCOCP [6]

Let $K \subseteq \mathbb{R}^2$ be a bounded open region, with boundary $\partial K$, and let $E = [0, T], 0 < T < \infty$ be a time space. The COCOCP governed by the VCNLHBVP, with control $\omega = \omega(\vec{x}, t) \in L^2(\rho)$ and state $\psi = \psi_0(\vec{x}, t) \in H^0_0(K)$, is

$$\psi_{tt} + \chi \psi + \vec{a}(\vec{x}, t)\psi = g(\vec{x}, \psi, t) + \omega - \omega_d, \quad \forall \vec{x}, t \in \rho = K \times E, \vec{x} = (y, z)$$  \hspace{1cm} (1)

With the conditions

$$\psi(\vec{x}, t) = 0, \forall \vec{x}, t \in \partial \rho = \partial K \times [0, T]$$  \hspace{1cm} (2)

$$\psi(\vec{x}, 0) = \psi^0(\vec{x}), \forall \vec{x} \in K$$  \hspace{1cm} (3)

$$\psi_t(\vec{x}, 0) = \psi^1(\vec{x}), \forall \vec{x} \in K$$  \hspace{1cm} (4)

where $\chi = \frac{1}{2} \sum_{\tau, q=1}^2 \partial_{\vec{y}_\tau} \left[ \vec{a}_{\tau q}(\vec{x}, t) \frac{\partial}{\partial \vec{y}_q} \right], \vec{a}(\vec{x}, t) \in L^\infty(K), (\forall \vec{y}, \tau = 1, 2)$ are positive arbitrary functions, $\omega_d = \omega_d(\vec{x}, t) \in L^2(\rho)$ is a desired control and $g \in L^2(\rho)$ is a given function.

The set of the admissible continuous classical controls is represented by $W_{ad}$, where $W_{ad} = \{ \omega \in L^2(\rho) | \omega(\vec{x}, t) \in U, \text{a.e. in } \rho \}$, with $U \subseteq \mathbb{R}^2$ is a convex and compact set.

The cost functional is defined by

$$G_0(\omega) = \int_0^T \left[ \frac{1}{2} (\psi - \psi_d)^2 + \frac{a}{2} (\omega - \omega_d)^2 \right] d\vec{x} dt$$  \hspace{1cm} (5)

where $\psi_d = \psi_d(\vec{x}, t) \in L^2(\rho)$ is the desired state.

The COCOCP is to obtain $\omega \in W_{ad}$ which minimizes equation (5).

Now, the weak form (WF) of the problems (1-4) for $\psi \in H^1_0(K)$ is formulated by

$$\langle \psi_{tt}, \varphi \rangle + B(t, \psi, \varphi) = (g(\vec{x}, \psi, t), \varphi)_K + (\omega, \varphi)_K - (\omega_d, \varphi)_K, \forall \varphi \in S = H^1_0(K)$$  \hspace{1cm} (6)

$$\langle \psi(0), \varphi \rangle = (\psi^0, \varphi), \forall \vec{x} \in K$$  \hspace{1cm} (7)

$$\langle \psi_t(0), \varphi \rangle = (\psi^1, \varphi), \forall \vec{x} \in K$$  \hspace{1cm} (8)

where $\psi^0 \in S, \psi^1 \in L^2(K)$ and the bilinear form is obtained by

$$B(t, \psi, \varphi) = \sum_{\tau, q=1}^2 \langle \vec{a}_{\tau q}(\vec{x}, t) \nabla \psi, \nabla \varphi \rangle_K + \langle \vec{a}(\vec{x}, t) \psi, \varphi \rangle_K$$

Assumption A: For each $\varphi, \psi \in S$ and $t \in E$, the following inequality is satisfied

$$|B(t, \psi, \varphi)| \leq \sigma_2 \| \psi \|_1 \| \varphi \|_1, \quad \text{where } \sigma_1 > 0$$

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(II) \(|B(t, \varphi, \varphi)| \geq \sigma_1 \| \varphi \|^2_2\), where \(\sigma_2 > 0\).

Now, suppose that \(\psi_t = \zeta\), then equalities (6-8) can be rewritten as follows
\[
(\zeta_t, \varphi) + B(t, \psi_t, \varphi) = (g(\tilde{x}, \psi_t, t), \varphi)_K + (\omega, \psi_t)_K - (\omega_d, \psi_t)_K, \forall \psi_t \in S
\]
\[
(\psi_t, \varphi) = (\zeta, \varphi)
\]
\[
(\psi_0, \varphi) = (\psi_0, \varphi), \forall \tilde{x} \in K
\]
\[
(\zeta_0, \varphi) = (\psi_0, \varphi), \forall \tilde{x} \in K
\]

3. Statement of the DSCOCP [6]: The COCOC is discretized by applying the GFEM as follows:
First, consider \((\tilde{x}, t)\) is dependent of \(t\), the region \(K\) can be divided into subregions (a polyhedron)
for every integer \((s), \) and \(E^s_j = \{t_j^s, t_{j+1}^s\}\) be a subdivision of the interval \(E\) and for \(j = 0, 1, ... , m - 1, \) where each
interval has a same length \((\Delta t = \frac{t}{m})\). Let \(S_s \subset S\) be the space of continuous piecewise affine
mapping (CPAM) in \(K\). The set of admissible discrete classical controls (DCC) is
\[
W^s_{ad} = \{\tilde{\omega} = \tilde{\omega}^s \in W^s_{ad} \mid \tilde{\omega}(\tilde{x}, t) = \tilde{\omega}_{ij} \in U^s i n \rho_{ij}, \}
\]
where \(\rho_{ij}: = \tilde{Z}_i^s \times E^s_j\)

Now, \(\forall \varphi \in S_s, \) and for \(j = 0, 1, ... , m - 1, \) the DWF of (9-12) can be obtained by
\[
\left(\zeta_j^s - \zeta_{j+1}^s, \varphi\right)_K + \Delta t B(\psi_{j+1}^s, \varphi) = \Delta t (g(t_j^s, \psi_{j+1}^s, \varphi))^2 + \Delta t (\omega_j^s - \omega_d(t_j^s))^2 \right)_K
\]
(13)
\[
(\psi_{j+1}^s - \psi_j^s, \varphi)_K = \Delta t (\zeta_{j+1}^s, \varphi)_K
\]
(14)
\[
(\zeta_0^s, \varphi)_K = (\zeta, \varphi)
\]
(15)
\[
(\zeta_j^s, \varphi)_K = (\psi_0, \varphi), \forall \tilde{x} \in K
\]
(16)
where \(\psi_j^s = \psi(t_j^s), j \in S_s\) for \(j = 0, 1, ... , m, \) \(\psi_0 \in S, \) and \(\psi_1 \in L^2(K)\).

The discrete cost functional (DCF) \(G_0^s(\omega^s)\) is given by
\[
G_0^s(\omega^s) = \Delta t \sum_{j=0}^{m-1} \int_K \left(\frac{1}{2}(\psi_{j+1}^s - \psi_d(t_{j+1}^s))^2 + \frac{\alpha}{2}(\omega_j^s - \omega_d(t_j^s))^2 \right) d\tilde{x}
\]
(17)

Hence, the DSCOC is to find \(\tilde{\omega}^s \in W_{ad}^s, \) such that
\[
G_0^s(\tilde{\omega}^s) = \min_{\omega^s \in W_{ad}^s} G_0^s(\omega^s)
\]

Assumption (B): (I) Suppose that the function \(g\) is defined on \(Z_i^s \times E_j^s \times \mathbb{R}\) continuous with respect to \(\psi_j^s\) and satisfies the following:
\[
|g(\tilde{x}, t_j^s, \psi_j^s)| \leq H_j(\tilde{x}) + \alpha|\psi_{j+1}^s|, \text{ for } j = 0, 1, ... , m - 1, \ H_j(\tilde{x}) = H(\tilde{x}, t_j^s) \in L^2(K) \text{ and } \alpha \geq 0.
\]

(II) \(|g(\tilde{x}, t_j^s, \psi_{j+1}^s) - g(\tilde{x}, t_j^s, \psi_j^s)| \leq C|\psi_{j+1}^s - \psi_j^s|, \text{ for } j = 0, 1, ... , m - 1, \) and \(C\) denotes the Lipschitz constant for any \(j\).

3.1 Theorem (Existence and uniqueness of the DWF) [9]: For any fixed \(j (0 \leq j \leq m - 1), \) and for fixed DCC \(\omega^s \in W_{ad}^s, \) the DWF (13-16) has a unique solution \(\psi_{\omega^s}^s = \psi^s = (\psi_0^s, \psi_1^s, ..., \psi_m^s), \) (for sufficiently small \(\Delta t\)).

4. Existence of the DSCOC:
The following assumptions are useful to study the existence of the DCC.
Assumption C: The cost functional is of Caratheodory type, and satisfies :
\[
\left|\frac{1}{2}(\psi_{j+1}^s - \psi_d(t_{j+1}^s))^2 + \frac{\alpha}{2}(\omega_j^s - \omega_d(t_j^s))^2 \right| \leq \gamma_j^s(\tilde{x}) + \theta_j(\psi_j^s)^2, \forall j = 0, 1, ... , m - 1
\]
where \(\gamma_j^s(\tilde{x}) = \gamma_j^s(\tilde{x}, t_j^s) \in L^2(K) \text{ and } \theta_j \geq 0, \forall j = 0, 1, ... , m - 1.\)

The proofs of the following theorem and lemmas are shown in the previous article [9].

4.1 Theorem: The operator \(\omega^s \mapsto \psi^s = \psi_{\omega^s}^s\) is continuous on \(L^2(K)\).

4.2 Lemma: The DCF is continuous with respect to (w.r.t.) the DCC on \(L^2(\rho)\).

4.3 Lemma: If the DCC \(\omega^s, \tilde{\omega}^s\) are bounded in \(L^2(\rho), \) and corresponding to the DSCCs \(\omega_j^s\) and \(\tilde{\omega}_j^s = \omega_j^s + \Delta \omega_j^s, \) then the discrete states are \(\psi_j^s, \tilde{\psi}_j^s = \psi_j^s + \Delta \psi_j^s, \) respectively, then \(\forall j = 0, 1, ... , m, \) :
\[
\|\Delta \psi_j^s\|_{L^2} \leq c \|\Delta \omega_j^s\|_{L^2} \text{ and } \|\Delta \tilde{\psi}_j^s\|_{L^2} \leq c \|\Delta \omega_j^s\|_{L^2}
\]
Or \(\|\Delta \psi_j^s\|_{L^2} \leq \tilde{c}, \) and \(\|\Delta \tilde{\psi}_j^s\|_{L^2} \leq \tilde{c},\)

5. The Necessary conditions for DCCOC:
The following theorem deals with the necessary conditions of the DCCOC governed by VCNLHBVP.
5.1 Theorem: Assume that DCF (17) is given and the DAWF (for the state equation) \( \eta^s_{\omega^s} = \eta^s = (\eta_0^s, \eta_1^s, ..., \eta_{m-1}^s) \) is given (for \( j = m - 1, m - 2, ..., 0 \)) by
\[
(\phi_{j+1}^s - \phi_j^s, \varphi)_K + \Delta t B(\eta_j^s, \varphi) = \Delta t \left( \eta_j^s g(\psi_{j+1}^s, (t_j^s, \psi_{j+1}^s), \varphi) \right)_K
\]
\[
\Delta t \left( \psi_{j+1}^s - \psi_d(t_{j+1}^s), \varphi \right)_K
\]
(18)

where \( \eta_j^s, \psi_j^s \in S_s \ (\forall j = 0, 1, ..., m) \). Then the Fréchet derivative of DCF is given by
\[
(D\eta^s_{\omega^s}, \omega^s - \omega^s) = \Delta t \sum_{j=0}^{m-1} \left( H_{\omega^s_j}^s(t_j^s, \psi_{j+1}^s, \eta_j^s, \omega_j^s), \Delta \omega_j^s \right)_K
\]
\[
= \Delta t \sum_{j=0}^{m-1} \left( \eta_j^s + \omega^s(t_j^s) - \omega_d(t_j^s), \Delta \omega_j^s \right)_K
\]
(21)

where \( \omega_j^s, \psi_j^s \in W^s_{ad}, \Delta \omega_j^s = \omega_j^s - \dot{\omega}_j^s \) for \( j = 0, 1, ..., m \), and \( H^s \) is called the Hamiltonian.

5.1 Corollary: The inequality
\[
\Delta t \sum_{j=0}^{m-1} \left( \eta_j^s + \omega(t_j^s) - \omega_d(t_j^s), \Delta \omega_j^s \right)_K \geq 0, \forall \omega_j^s \in W^s_{ad}
\]
(22)
is equivalent with the minimum principle blockwise \( \forall j = 0, 1, ..., m - 1 \)
\[
\left( \eta_j^s + \omega(t_j^s) - \omega_d(t_j^s), \omega_j^s \right)_{T_i} = \min_{\omega^s \in W^s_{ad}} \left( \eta_j^s + \omega(t_j^s) - \omega_d(t_j^s), \omega_j^s \right)_{T_i}
\]
(23)

6. Main results (Solution methods):
This section is devoted to present our method which is used to solve the DCCOC governed by the VCNLHBVP, the DWF (and the DAWF) are solved by using the mixed GFEM-IFDS, while the minimum values for the DCF and the DCOC are found by using, separately, each one of the GM, FWM, or GPM. Within each of these three methods, the ARSO and the OPSO are used, separately, to improve the value of the DCOC. The following algorithm shows the steps of this method in details.

6.1 ALGORITHM: Let \( \omega \in (0, 1) \), \( \delta^s \) be a sequence with \( \delta^s \in (0, \infty) \), or \( \delta^s \in (0, 1] \), for each \( s \). \( \mu > 0 \), and let \( \omega^s \in U \) be an initial control.

Step1: Set \( s = 0 \).

Step 2: Solve the DWF (13-16) (the DAWF (18-20)) by using GFEM-IFDS to get the state \( \psi^s \) (the adjoint solution \( \eta^s \)), and then calculate \( G(\omega^s) \) and \( DG(\omega^s) \) from (17) and (21), respectively.

Step 3: Find a new direction (new control) \( u^s \in U \) (i.e. a direction \( u^s - \omega^s \)), by applying the following methods, separately:

(a) GM: Find \( u^s \in U \), such that:
\[
u^s = \omega^s - \frac{1}{\mu} DG(\omega^s)
\]
(b) FWM: Find \( u^s \in U \), such that: \( DG(\omega^s), u^s - \omega^s = \min_{\omega^s \in U} (DG(\omega^s), u^s - \omega^s) \)
(c) GPM: Find \( u^s \in U \), such that:
\[
\xi^s = (DG(\omega^s), u^s - \omega^s) + \frac{\mu}{2} \| u^s - \omega^s \|^2 = \min_{\omega^s \in U} (DG(\omega^s), u^s - \omega^s) + \frac{\mu}{2} \| u^s - \omega^s \|^2
\]

Step 4: Solve the DWF (13-16) to find the state solution \( \psi^s \) corresponding to the new \( u^s \).

Step 5: Calculate \( \xi^s = (DG(\omega^s), u^s - \omega^s) \), (\( \xi^s = -\frac{1}{\mu} \| DG(\omega^s) \| Z \) in the GM)

If \( \xi^s = 0 \), then stop the process \( \omega \) (where \( \| \| \) is the norm-2 with respect to vector space \( Q \)).

Step 6: Choose \( \delta^s \) by using one of the following methods:

ARSO: Assume an initial value \( \delta^s \in [0, +\infty) \) (or \( \delta^s \in [0, 1] \)). If \( \delta^s \) satisfies the inequality
\[
X_\delta(\delta^s) = G(\omega^s + \delta^s(u^s - \omega^s)) - G(\omega^s) \leq \delta^s b \xi^s
\]
we set \( \delta := \delta/c \), and choose the last \( \delta \in (0, \infty) \) for \( \delta^s \), that satisfies the above inequality. If not satisfied, we denote \( \delta := \delta/c \), and choose for \( \delta^s \) the first \( \delta^s \in (0, \infty) \) (or \( \delta^s \in (0, 1] \) in GM) that satisfies the above inequality.

OPSO: Find an \( \delta^s \in [0, 1] \), such that:
\[
(\omega^s + \delta^s(u^s - \omega^s), \min_{\delta^s \in [0, 1]} (DG(\omega^s), u^s - \omega^s)
\]

Step 7: Set \( \omega^{s+1} = \omega^s + \delta(u^s - \omega^s) \), \( s = s + 1 \) and we go to step 2.

7. Numerical examples
This section contains some illustrative examples which show the activity of the methods which are
given in algorithm (6.1). MATLAB software is used to achieve the above algorithm. The GFEM-IFDS is used in step (2) to find the DS $\psi^s(\eta)^s$, with $n = 9 \ (h = 0.1), \ m = 20. \ (\Delta t = 0.05)$. In the GM, GPM and FWM, the parameters take the values of $b = c = 0.5$ and $\mu = 0.5$.

7.1 Example: Consider the following COCOCPP governed by the VCNLHBVP:

$$\psi_{tt} - \frac{\partial}{\partial y} \left[ e^x \frac{\partial \psi}{\partial y} \right] - \frac{\partial}{\partial z} \left[ e^y \frac{\partial \psi}{\partial z} \right] + e^{ny-z} \psi = g(\bar{x}, t) + \sin(\psi) - \sin(\psi_d) + \omega - \omega_d, \quad \forall (\bar{x}, t) \in \rho,$$

$$E \equiv [0,1], \ K = [0,1] \times [0,1], \text{ and } \bar{x} = (y, z).$$

$\psi(\bar{x}, 0) = 0$, and 

$$\psi_t(\bar{x}, 0) = 0.75yz(1 - e^{x(1-x)}) \ (1 - y) \quad \forall \bar{x} \in K.$$

where

$$g(\bar{x}, t) = \frac{3 \ x \ e^x \sin(t)(e^{-x(2z-1)})^2 - e^y \left(2y e^{-x(2z-1)} \sin(t)(2z - 1) \left(\frac{3y}{4} - \frac{3}{4}\right) + 2yz e^{-x(2z-1)} \sin(t)(2z - 1)^2 \left(\frac{3y}{4} - \frac{3}{4}\right)ight) + yz \sin(t) \left(\frac{3y}{4} - \frac{3}{4}\right)}{2z} \left( e^{-x(2z-1)} - 1 \right) \left( yze^{-ny-z} \sin(t)((3y)/4 - 3/4)(e^{-x(2z-1)} - 1) \right)$$

The control constraint is $U = [-0.5, 1]$ and the cost function (5) with

$$\psi_d(\bar{x}, t) = 0.75yz(1 - y)(1 - e^{x(1-x)}) \sin(-t), \quad \forall (\bar{x}, t) \in \rho,$$

and

$$\omega_d(\bar{x}, t) = \left\{ \begin{array}{ll}
0.25, & \text{for } 0 \leq t < 0.5 \\
0.25, & \text{for } 0.5 \leq t \leq 1
\end{array} \right.$$

with the initial control

$$\omega_0(\bar{x}, t) = 0.35 - (2t e^{-t}), \quad \forall (\bar{x}, t) \in \rho.$$

**First**, depending on the above initial control and its corresponding state, the following results are obtained according to the optimization methods with ARSO .

(1) In the GM: the optimal control and the corresponding state are obtained after 11 iterations. The results are: $G_0(\omega^*)=4.6678e-06, \ g_0^s = 9.5e-03$, and $e_0^s=1.1e-03$

where $g_0^s$ and $e_0^s$ are the discrete maximum errors for the state and control, respectively.

The optimal control and its corresponding state are obtained at $t = 0.5$ by the following figures.

![Figure 1- The optimal control](image1)

![Figure 2- The Corresponding state](image2)

(II) In the FWM: the optimal control and corresponding state are showed after 117 iterations, the results are: $G_0(\omega^*)=7.0095e-06, \ g_0^s = 9.8e-03$, and $e_0^s=9.9e-03$

The following figures show the optimal control and its corresponding state at $t = 0.5$:

![Figure 3- The optimal control](image3)

![Figure 4- The Corresponding state](image4)
(III) In the GPM: the optimal control and corresponding state are obtained after 4 iterations. The results are: $g_0(\omega^s)=4.6450e-06, \varphi_s=9.5e-03,$ and $\xi_s=7.7140e-04$
The following figures of this method are also shown at $t = 0.5$.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure5.png}
\caption{The optimal control}
\end{figure}
\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure6.png}
\caption{The Corresponding state}
\end{figure}

\textbf{Second}, the following results are obtained by using the optimization methods with \textbf{OPSO}:
(I) In the GM and GPM: the optimal control and corresponding state are given after 3 iterations
The results in this case are: $g_0(\omega^s)=4.5746e-06, \varphi_s=9.5e-03,$ and $\xi_s=1.5e-03$.
The optimal control and its corresponding state are shown at $t = 0.5$ by the following figures.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure7.png}
\caption{The optimal control}
\end{figure}
\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure8.png}
\caption{The Corresponding state}
\end{figure}

(II) In the FWM: the optimal control and corresponding state are given after 60 iterations. The results are: $g_0(\omega^s)=7.8110e-06, \varphi_s=9.7e-03,$ and $\xi_s=9.8e-03$.
The following figures show the optimal control and its corresponding state at $t = 0.5$.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure9.png}
\caption{The optimal control}
\end{figure}
\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure10.png}
\caption{The Corresponding state}
\end{figure}

\textbf{7.2 Example}: Consider the following CCOCP governed by the VCNLHBVP:
\[
\psi_{tt} - \frac{\partial}{\partial y} \left( y + 1 \frac{\partial \psi}{\partial y} \right) - \frac{\partial}{\partial z} \left( z + 1 \frac{\partial \psi}{\partial z} \right) + \psi = g(\vec{x}, t) + \sin(\psi) - \sin(\psi_d) + \omega - \omega_d, \quad \forall (\vec{x}, t) \in \rho, E = [0,1], \ K = [0,1] \times [0,1], \ \vec{x} = (y, z)
\]
ψ(\(\vec{x}, t\)) = 0, \(\forall (\vec{x}, t) \in \partial Q = \partial K \times [0, T]\).
ψ(\(\vec{x}, 0\)) = 0, and \(ψ_1(\vec{x}, 0) = yz(1 - y)(1 - z), \forall \vec{x} \in K\)
where \(g(\vec{x}, t) = y^2 \sin(t)(y - 1)(z - 1) - y z \sin(t)(y - 1)(z - 1) - yz \sin(t)(y - 1)(z + 1) - z \sin(t)(y - 1)(z - 1) - 2z \sin(t)(y + 1)(z - 1) - yz \sin(t)(y - 1) - yz \sin(t)(y - 1)(z - 1)\)
The control constraint is \(U = [-6, 5]\) and the cost functional (5), with
\(ψ_d(\vec{x}, t) = yz(1 - y)(1 - z) \sin(t), \forall (\vec{x}, t) \in ρ, and \)
\(ω_d(\vec{x}, t) = \begin{cases} 
3.25, & \text{for } 0 \leq t < 0.5 \\
-3, & \text{for } 0.5 \leq t < 1
\end{cases} \)
with the initial control
\(ω_0(\vec{x}, t) = \begin{cases} 
3.35, & 0 \leq t < 0.2 \\
2.25, & 0.2 \leq t < 0.5 \\
-1, & 0.5 \leq t < 0.7, \forall (\vec{x}, t) \in ρ.
\end{cases} \)
First, depending on the above initial control and its corresponding state, the following results are obtained according to the optimization methods with ARSO.
(I) In the GM: the optimal control and corresponding state are obtained after 10 iterations. The results are: \(G_0(\omega^s) = 6.6397e-06, q_s = 5.8e-03, \) and \(ε_s = 5.5014e-04.\)
The optimal control and its corresponding state at \(t = 0.5\) are shown by the following figure.

![Figure 11- The optimal control](image)
![Figure 12- The Corresponding state](image)

(II) In the FWM: the optimal control and corresponding state are obtained after 80 iterations. The results are: \(G_0(\omega^s) = 7.8843e-06, q_s = 6.1e-03, \) and \(ε_s = 4.1e-03.\)
The following figures show the optimal control and its corresponding state.

![Figure 13- The optimal control](image)
![Figure 14- The Corresponding state](image)

(III) In the GPM: the optimal control and corresponding state are obtained after 7 iterations The results are: \(G_0(\omega^s) = 6.3639e-06, q_s = 5.8e-03, \) and \(ε_s = 3.9563e-04.\)
The following figures show the optimal control and its corresponding state.
Second, the following results are obtained by using the optimization methods with OPSO:

(I) In the GM and GPM: the optimal control and corresponding state are given after 4 iterations, with the results in this case are: $G_0(ω^5)=6.6890e-06$, $q_5=5.7e-03$, and $ɛ_5=3.6205e-04$.

The optimal control and its corresponding state are shown by the following figures.

(II) In the FWM: the optimal control and corresponding state are given after 42 iterations. The results are: $G_0(ω^5)=1.0879e-05$, $q_5=6.7e-03$, and $ɛ_5=9.0e-03$.

The following figures show the optimal control and its corresponding state at $t = 0.5$.

Conclusions: The GFEM-IFDS is applied to solve the DWF of VCNLHBVP as well as the DAWF to find the NS, using the step length of the space variable $h = 0.1$ and step length of the time variable $Δt = 0.05$, along with the parameters ($b = 0.5$, $c = 0.5$ and $μ = 0.5$) in the ARSO. From the results of the two examples above we can point out the following conclusions:

(I) The GFEM-IFDS (according to the use of the values $h$ and $Δt$) is an accurate, appropriate and fast method to solve the DWF and DAWF.
(II) The results obtained by applying the GPM with ARSO are better than those obtained by using the GM. On the other hand, the results obtained by these two methods are better and faster than those obtained by applying the FWM with ARSO.

(III) The results obtained by the GPM and GM with OPSO method are better and faster than those obtained by using the FWM with the OPSO method.

(IV) For the OPSO and the ARSO, which were used inside the GPM, although the GM method was faster than the FWM one, it cannot be used in general, since it is only suitable for a quadratic cost function.

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References