On Existence and Uniqueness of an Integrable Solution for a Fractional Volterra Integral Equation on $R^+$

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Abstract

In this paper, by using the Banach fixed point theorem, we prove the existence and uniqueness theorem of a fractional Volterra integral equation in the space of Lebesgue integrable $L_1(R^+)$ on unbounded interval $[0, \infty)$.

Keywords: space of Lebesgue integrable - Banach fixed point theorem - fractional Volterra integral equation - Superposition Operator.

1. Introduction

Since the last century, time-dependent problems of non-linear differential equations and integral equation have been studied by many authors; see [1-7].

The subject of nonlinear fractional integral equation considered as an important branch of mathematics because it is used for solving in many fields such as physics, engineering and economics [1-4].

In this paper, we will prove the existence and uniqueness theorem of a fractional Volterra integral equation in the space of Lebesgue integrable $L_1(R^+)$ on unbounded interval $[0, \infty)$ of the type:

$$x(t) = g(t) f(t, x(t)) + h(t) + \int_0^t \frac{e^{-(t-s)(t-s)^{\alpha-1}}}{\Gamma(\alpha)} f(s, x(s))ds,$$

where $0 < \alpha < 1, \ t > 0$.

2. Preliminaries

Let $R$ be the field of real number, $R^+$ be the interval $[0, \infty)$. If $A$ is a Lebesgue measurable subset of $R$, then the symbol $\text{mas}(A)$ stands for the Lebesgue measure of $A$.

Further, we denote by $L_1(A)$ the space of all real functions, defined and Lebesgue measurable on the set $A$. The norm of a function $x \in L_1(A)$ is defined in the standard way by the formula,

$$\|x\| = \|L_1(A)\| = \int_A |x(t)| \, dt$$

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Obviously, $L_1(A)$ forms a Banach space under this norm. The space $L_1(A)$ is called the Lebesgue space. In the case when $A = R^+$, we write $L_1$ instead of $L_1(R^+)$. One of the most important operators studied in the nonlinear functional analysis is the so-called superposition operator [8]. Now, let us assume that $A \subset R$ is a given interval bounded. 

**Definition 2.1** [8]: Assume that a function $f(t,x) = f: I \times R \to R$ satisfies the so-called Carathéodory condition, i.e. it is measurable in $t$ for any $x \in R$ and continuous in $x$ for almost all $t \in I$. Then to every function $x = x(t)$ which is measurable on $I$ we may assign the function $(Fx)(t) = f(t, x(t))$, $t \in I$. The operator $F$ defined in such a way is said to be the **superposition operator** generated by the function $f$.

**Theorem 2.1** [9]: The superposition operator $F$ generated by a function $f$ maps continuously the space $L^1(I)$ into itself if and only if $|f(t,x)| \leq a(t) + b|x|$ for all $t \in I$ and $x \in R$, where $a(t)$ is a function from $L^1(I)$ and $b$ is a nonnegative constant.

This theorem was proved by Krasnoselskii [9] in the case when $I$ is a bounded interval. The generalization to the case of an unbounded interval $I$ was given by Appell and Zabrejko [8].

**Definition 2.2** [10]: A function $f : A \to R^m$, $A \subset R^n$ is said to be Lipschitz continuous if there exists a constant $L$, $L > 0$ (called the Lipschitz constant of $f$ on $A$) such that $|f(x) - f(y)| \leq L|x - y|$, for all $x, y \in A$.

**Definition 2.3** [11]: Let $(X,d)$ be a metric space and $T : X \to X$ is called contraction mapping, if there exist a number $\gamma < 1$, such that: $d(Tx,Ty) \leq \gamma d(x,y)$, $\forall x,y \in X$.

**Theorem 2.2** [12]: Let $X$ be a closed subset of a Banach space $E$ and $T : X \to X$ be a contractive, then $T$ has a unique fixed point.

**Definition 2.4** [13]: Let $[a,b]$ $(\infty < a < b < \infty)$ be a finite interval on the real axis $R$, the Riemann-Liouville fractional integral $I_{a+}^f$ of the order $\alpha \in C(\mathbb{R}(\alpha) > 0)$ is define by:

$$I_{a+}^f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)ds}{(t-s)^{1-\alpha}} \quad (t > a; \quad \mathbb{R}(\alpha) > 0).$$

**Definition 2.5** [14]: Let $[a,b]$ $(\infty < a < b < \infty)$ be a finite interval on the real axis $R$, the Riemann-Liouville fractional integral $D_{a+}^f$ of the order $\alpha \in C(\mathbb{R}(\alpha) \geq 0)$ is define by:

$$D_{a+}^f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)ds}{(t-s)^{1-n+\alpha}} \quad (t > a ; n = [\mathbb{R}(\alpha)] + 1)$$

where $[\mathbb{R}(\alpha)]$ denotes the integral part of $\mathbb{R}(\alpha)$. "i.e. $[\mathbb{R}(\alpha)]$ it satisfies $\mathbb{R}(\alpha) \leq [\mathbb{R}(\alpha)] \leq [\mathbb{R}(\alpha)] + 1."

3. Existence Theorem

Define the operator $H$ associated with integral equation (1.1) which takes the following form:

$$Hx = Ax + Bx.$$  \hspace{1cm} (3.1)

where $(Ax)(t) = g(t) f(t, x(t))$, $(Bx)(t) = h(t) + \int_0^t e^{-\alpha(t-s)} f(s, x(s))ds = h(t) + KFx(t)$, $\quad$ (3.2)

where, $(Kx)(t) = \int_0^t e^{-\alpha(t-s)} x(s)ds$, $Fx = f(t, x)$ are linear operators at superposition, respectively.

We shall treat the equation (3.1) under the following assumptions. Assume that:

i) $g : R^+ \to R$ is a bounded function such that : $M = \sup_{t \in R^+} |g(t)|$,

and $h : R^+ \to R$, such that $h \in L_1(R^+)$. ii) $f : R^+ \times R \to R$ satisfies Lipschitz condition with positive constant $L$ such that $|f(t, x(t)) - f(t, y(t))| \leq L|x(t) - y(t)|$, for all $t \in R^+$. iii) $LM + L < 1$.

Now, for the existence of a unique solution of our equation, we need to prove the following theorem:

**Theorem 3.1**: If the assumptions (i)-(iii) are satisfied, then the equation (1.1) has a unique solution, where $x \in L_1(R^+)$. 

**Proof**: Firstly, we will prove that $H : L_1(R^+) \to L_1(R^+)$. 

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Secondly, will prove that $H$ is a contraction.

Consider the operator $H$ as:

$$Hx(t) = g(t)f(t, x(t)) + h(t) + \int_0^t e^{-(t-s)(s-a)} \frac{f(s, x(s))}{\Gamma(a)} ds$$

Then our equation (1.1) becomes

$$x(t) = Hx(t).$$

We notice that by assumption (ii), we have

$$|f(t, x)| = |f(t, 0) - f(t, 0) + f(t, 0)|
\leq |f(t, x) - f(t, 0)| + |f(t, 0)|
\leq L |x - 0| + |f(t, 0)|
\leq L |x| + a(t)$$

Where $|f(t, 0)| = a(t)$

To prove that $H : L_1 (R^+) \to L_1 (R^+)$.

Let $x \in L_1 (R^+)$,

then we have

$$\int_0^\infty |Hx(t)| dt = \int_0^\infty |g(t)f(t, x(t)) + h(t) + \int_0^t e^{-(t-s)(s-a)} \frac{f(s, x(s))}{\Gamma(a)} ds| dt$$

$$\leq \int_0^\infty |g(t)||f(t, x(t))| dt + \int_0^\infty |h(t)| dt + \int_0^\infty \left| \int_0^t e^{-(t-s)(s-a)} \frac{f(s, x(s))}{\Gamma(a)} ds \right| dt$$

$$\leq M \int_0^\infty |a(t) + L|x(t)|| dt + \|h\|
\leq M |a| + L\|x\| + \|h\|$$

Let $J = \int_s^\infty e^{-(t-s)(t-s-a)} \frac{f(s, x(s))}{\Gamma(a)} dt$

then:

$$J = \int_s^\infty \frac{e^{-(t-s)(t-s-a)}}{\Gamma(a)} dt = \frac{1}{\Gamma(a)} \int_0^\infty e^{-x} \Gamma(a-1) dx = \frac{\Gamma(a)}{\Gamma(a)} = 1$$

Then, we get

$$\int_0^\infty |Hx(t)| dt \leq M \int_0^\infty |a(t) + L|x(t)|| dt + \|h\| + \int_0^\infty |f(s, x(s))| ds$$

$$\leq M |a| + LM \int_0^\infty |x(t)|| dt + \|h\| + \int_0^\infty |a(s) + L|x(s)|| ds$$

$$\leq M |a| + LM \||x\| + \|h\| + |a| + L\|x\|$$

Then

$$H : L_1 (R^+) \to L_1 (R^+).$$

Secondly, we prove that $H$ is a contraction.

Let $x, y \in L_1 [0, \infty)$, then

$$\int_0^\infty |Hx(t) - Hy(t)| dt = \int_0^\infty \left| g(t)f(t, x(t)) + h(t) + \int_0^t e^{-(t-s)(t-s-a)} \frac{f(s, x(s))}{\Gamma(a)} ds \right| dt$$

$$- g(t)f(t, y(t)) - h(t) - \int_0^t e^{-(t-s)(t-s-a)} \frac{f(s, y(s))}{\Gamma(a)} ds| dt$$

$$\leq \int_0^\infty |g(t)||f(t, x(t)) - f(t, y(t))| dt + \int_0^\infty |h(t)| dt$$

$$+ \int_0^\infty \left| \int_0^t e^{-(t-s)(t-s-a)} \frac{f(s, x(s))}{\Gamma(a)} ds \right| dt$$

$$\leq M \int_0^\infty \|x(t) - y(t)\| dt + \int_0^\infty \left| \int_0^t e^{-(t-s)(t-s-a)} \frac{f(s, x(s))}{\Gamma(a)} ds \right| dt$$

$$\leq LM \|x - y\| + \int_0^\infty \left| \int_0^t e^{-(t-s)(t-s-a)} \frac{f(s, x(s))}{\Gamma(a)} ds \right| dt$$

$$\leq LM \|x - y\| + \int_0^\infty \|x(s) - y(s)\| ds$$
\[ \leq LM ||x - y|| + L \int_{0}^{\infty} |x(s) - y(s)| \, ds \]
\[ \leq LM ||x - y|| + L ||x - y|| \leq [LM + L] ||x - y|| \]

Hence, by using Banach fixed point theorem, 
\( H \) has a unique point, which is the solution of the equation (1.1), where \( x \in L_1 [0, \infty) \).

**Conclusion**

In this paper, by using Banach fixed point theorem we proved the existence and uniqueness theorem of a fractional nonlinear Volterra integral equation in the space of Lebesgue integrable \( L_1(R^+) \) on unbounded interval \([0, \infty)\) of the type:

\[ x(t) = g(t) f(t, x(t)) + h(t) + \int_{0}^{t} \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x(s)\right) \, ds, \]

where \( 0 < \alpha < 1, \, t > 0 \). under the following assumptions:

i) \( g : R^+ \rightarrow R \) is a bounded function such that: \( M = \sup_{t \in R^+} |g(t)| \),

and \( h : R^+ \rightarrow R \), such that \( h \in L_1(R^+) \).

ii) \( f : R^+ \times R \rightarrow R \) satisfies Lipschitz condition with positive constant \( L \) such that

\[ |f(t, x(t)) - f(t, y(t))| \leq L|x(t) - y(t)|, \quad \text{for all } t \in R^+. \]

iii) \( LM + L < 1 \).

**References**


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