The Frequency of t-Practical Numbers

Saad A. Baddai
Department of Mathematics, College of Science for Women University of Baghdad, Iraq

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Abstract
Hausman and Shapiro gave an estimate for the number of practical numbers \( n, n \leq x \) to be
\[
O \left( \frac{x}{(\log x)^\beta} \right),
\]
for every positive \( \beta < \frac{1}{2} \left( \frac{1}{\log 2} - 1 \right)^2 \). In this paper, we generalize Hausman and Shapiro bound by proving the number of t-practical numbers \( n, (n \leq x), (t \geq 1) \) to be
\[
O \left( \frac{x}{(\log x)^\beta} \right),
\]
for every positive \( \beta < \frac{1}{2} \left( \frac{1}{\log 2} - 1 \right)^2 \), and \( 1 \leq t \leq \exp \left( (\log x)^{\delta_1} \right) \) for any \( \delta_1 \) satisfying \( 0 < \delta_1 < 1 - (1 + \sqrt{2} \beta) \log 2 \). We mean by the t-practical number \( n \), the number in which every integer \( 1 \leq m \leq tn \) is of the form
\[
m = \sum_{d|m} c_d d, \quad 0 \leq c_d \leq t
\]

Keywords: Bound for the t-practical numbers, Number of t-practical numbers

(\( t \geq 1 \), 1)

سعد عابد بداي
قسم الرياضيات، كلية العلوم للبنات، جامعة بغداد، بغداد، العراق

الخلاصة:
في هذا البحث تم حساب تكرر الاعداد العملية ذات التكرار \( (t \geq 1), t \) ذات التكرار \( n \leq x, n \) للفترة الثابتة \( 0 < \delta_1 < 1 - (1 + \sqrt{2} \beta) \log 2 \) ونأتي \( \delta_1 \) في الفترة

\[
0 < t \leq \exp \left( (\log x)^{\delta_1} \right)
\]

ولأي \( \delta_1 \) في الفترة

\[
0 < \delta_1 < 1 - (1 + \sqrt{2} \beta) \log 2 \]

وهذه النتيجة تغطي كافة الاعداد العملية \( n \leq x, n \) بكرار (\( t \geq 1), t \) بكرار في البحث. شابور، هاوسمان

*Email: baddai1955@gmail.com
1. Introduction

The \( t \)-practical number is a generalization of practical numbers, \( n \), defined in an earlier work [1], where Margenstern conjectured that the number of \( t \)-practical numbers \( n, n \leq x \) is denoted by \( P(x) \) and

\[
P(x) \sim \lambda \frac{x}{\log x}
\]

with \( \lambda \approx 1.341 \). Further results related to \( P(x) \) were given by Weingartner [2], who showed that

\[
P(x) = \frac{cx}{\log x} \{ 1 + O\left( \frac{\log \log x}{\log x} \right) \}
\]

for a positive constant \( c \) and \( x \geq 3 \). These results were demonstrated by Wang and Sun [3] to be

\[
P(x) \sim \frac{cx}{\log x}
\]

as \( x \to \infty \).

In this paper, our generalization covered all \( t \)-practical numbers and the case when \( t=1 \) implies the number of practical numbers given by Hausman and Shapiro, which was established in [4].

2. Preliminary Results and Definitions

**Definition (2.1) [1]:** Let \( n \geq 1 \). Then \( n \) is called \( t \)-practical number if every integer \( v \) \( m \in 1 \leq m < n \) is having the form

\[
m = \sum_{d|n} c_d d, \quad c_d = 0, 1.
\]

**Definition (2.2):** The integer \( n, n \geq 1 \) is called \( t \)-practical number if every integer \( m, 1 \leq m \leq tn, (t \geq 1) \) is written as

\[
m = \sum_{d|n} c_d d, \quad (1 \leq c_d \leq t).
\]

**Definition (2.3) [5]:** The function \( \omega (n) \) is defined to be the number of distinct prime factors of distinct prime factors of \( n \), where \( \omega (n) = r \), when \( n = p_1^{a_1} \ldots p_r^{a_r} \)

The following Lemmas will be required.

**Lemma (2.1) [4]:** For \( \omega (n) \) and any \( \epsilon > 0 \) (possibly a function of \( x \)), the number of \( n \leq x \) such that \( \omega (n) > (1 + \epsilon) \log \log x \) is

\[
O\left( \frac{x}{(\log x)^{\epsilon/2}} \right)
\]

The function \( O \) is uniform in \( \epsilon \).

**Lemma (2.2):** Suppose that \( n = n^*, p_1 \ldots p_i, (p_i, n^*) = 1 \) and \( p_i \) are distinct primes, then if for \( i = 1, 2, \ldots, l, p_i \leq t\sigma (n^*, p_1, \ldots p_{i-1}) + 1 \) then

\[
p_i \leq [t\sigma (n^*) + i]^{2^{i-1}} \quad \ldots (1)
\]

(\( \sigma (n) \) is the sum of divisors of \( n \)).

**Proof:** We have that

\[
p_i \leq t\sigma (n^* p_1 \ldots p_{i-1}) + 1 \quad \ldots (2)
\]

We shall proceed by induction on \( i \) to show that

\[
t\sigma (n^* p_1 \ldots p_{i-1}) \leq [t\sigma (n^*) + i - 1]^{2^{i-1}} \quad \ldots (3)
\]

for \( i = 1 \), then (3) is true. As an induction hypothesis, assume that (3) is true for \( i - 1 \). Then we write

\[
t\sigma (n^* p_1 \ldots p_i) = t\sigma (n^* p_1 \ldots p_{i-1}) (p_i + 1)
\]

\[
\leq [t\sigma (n^* p_1 \ldots p_{i-1})] [t\sigma (n^* p_1 \ldots p_{i-1}) + 2]
\]

and by the induction hypothesis, we have

\[
t\sigma (n^* p_1 \ldots p_i) \leq [t\sigma (n^*) + i - 1]^{2^{i-1}} [(t\sigma (n^*) + i - 1)^{2^{i-1}} + 2]
\]

\[
t\sigma (n^* p_1 \ldots p_i) \leq [t\sigma (n^*) + i - 1]^{2^i} + 2[t\sigma (n^*) + i - 1]^{2^{i-1}} + 1
\]

\[
t\sigma (n^* p_1 \ldots p_i) \leq (t\sigma (n^*) + i)^{2^i}
\]

Therefore, inequality (3) is true for \( i \). Using (2) and (3), we get

\[
p_i \leq t\sigma (n^* p_1 \ldots p_{i-1}) + 1 \leq [t\sigma (n^*) + i - 1]^{2^{i-1}} + 1
\]

Then,

\[
p_i \leq [t\sigma (n^*) + i]^{2^{i-1}}
\]
This is the required solution.

3. The Frequency of t-Practical Numbers

The following results were proved by Hausman and Shapiro [4].

**Theorem (3.1) [4]:** the number of practical numbers n less than or equal to x is

\[ O \left( \frac{x}{(\log x)^{\beta}} \right) \]

for every fixed positive \( \beta < \frac{1}{2} \left( \frac{1}{\log 2} - 1 \right)^2 \).

**Lemma (3.1) [4]:** For \( \omega(n) \) equal to the number of distinct prime factors of n and any given \( \varepsilon > 0 \) (possibly function of x), the number of \( n \leq x \), such that \( \omega(n) > (1 + \varepsilon) \log \log x \), is

\[ O \left( \frac{x}{(\log x)^{\varepsilon/2}} \right) \]

where the \( O \) uniform in \( \varepsilon \).

A generalization of theorem (3.1) [4] is the substance of the following main theorem.

**Theorem (3.2):** Let \( t \geq 1 \) and \( 0 < \beta < \frac{1}{2} \left( \frac{1}{\log 2} - 1 \right)^2 \). Then the number of t-practical numbers \( n \leq x \) is

\[ O \left( \frac{x}{(\log x)^{\beta}} \right) \]

This estimate is a uniform for \( t \) in the range

\[ 1 \leq t \leq \exp \left( (\log x)^{\delta_1} \right) \]

for any \( \delta_1 \) with \( 0 < \delta_1 < 1 - (1 + \sqrt{2\beta}) \log 2 \).

**Proof:** Let \( 0 < \varepsilon < \frac{1}{(\log 2) - 1} \). Then form Lemma (3.1) [4], the number of t-practical numbers \( n \leq x \) is

\[ O \left( \frac{x}{(\log x)^{\varepsilon/2}} \right) \]

Now, we consider the t-practical numbers \( n \leq x \) with \( \omega(n) \leq (1 + \varepsilon) \log \log x \) and \( n \) to have the form

\[ q_1 \ldots q_k \cdot p_1 \ldots p_l \]

where \( q_j, p_i \) are distinct primes \( p_i \neq q_j \), for \( 1 \leq i \leq l \), \( 1 \leq j \leq k \)

and

\[ q_j \leq t + 1 < p_1 < \ldots < p_l \]

Since \( \omega(n) = l + k \), then

\[ (l + k) \leq (1 + \varepsilon) \log \log x \]

By setting \( n^* = q_1 \ldots q_k \), then from (1) we have

\[ n^* \leq (t + 1)^{(1 + \varepsilon) \log \log x} \]

or

\[ n^* \leq \exp \left( (2 \log \log x) \log(t + 1) \right) \]

Therefore the t-practical numbers are of the form

\[ n^* p_1 \ldots p_l \]

with \( l < (1 + \varepsilon) \log \log x \) and \( p_l \) larger than primes dividing \( n^* \). From Lemma (2.2), we have

\[ p_l \leq t \sigma(n^*) + p_{l-1} + 1 \]

and

\[ p_l \leq \left[ t \sigma(n^*) + l \right]^{2^{l-1}} \]

From [5], we have

\[ \sigma(n^*) = O(n^* \log \log n) \]

Then from (3), we get

\[ \sigma(n^*) = O(n^* \log \log n) \leq \exp \left( c_1 \log \log x \right) \log(t + 1) \]

where \( c_1 > 0 \) is a constant. Therefore (6) and (7) imply that

\[ p_l \leq \left[ t \exp \left( c_1 \log \log x \right) \log(t + 1) + l \right]^{2^{l-1}} \]

Since \( i \leq l(1 + \varepsilon) \log \log x \), then (8) becomes

\[ p_l \leq \left[ t \exp \left( c_2 \log \log x \right) \log(t + 1) \right]^{2^{l-1}} \]
and $c_2 > 0$ is a constant. From (3) and (9), the number of $t$-practical numbers having the form (4) is at most,

$$[\exp \{ (2 \log \log x). \log(t + 1) \}]. [\exp \{ c_2 \log \log x \}. \log(t + 1)]^{1+2+\cdots+2^{l-1}}$$

$$[\exp \{ \log(x) \log(t + 1) \}]\cdot [\exp \{ c_2 \log \log x \}. \log(t + 1)]^{2l-1}$$

$$\leq [\exp \{ c_3 \log \log x \}. \log(t + 1)]^{2^{l}(1+\varepsilon) \log \log x + 1}. t^{2^{l}(1+\varepsilon) \log \log x}$$

... (10)

where $c_3 > 0$ is a constant. Since,

$$2^{1+\varepsilon} \log \log x = e^{\log(2 \log \log x) + \varepsilon} = e^{\log(2 \log \log x) + (1+\varepsilon) \log 2}$$

therefore

$$2^{(1+\varepsilon) \log \log x} = (\log x)^{(1+\varepsilon) \log 2}$$

... (11)

and the number of $t$-practical numbers given in (10) is at most

$$[\exp \{ \log(x) \log(t + 1) \}]^{1+\log \log x + (1+\varepsilon) \log 2}. t^{(1+\varepsilon) \log 2 - 1}$$

... (12)

where

$$t^{(1+\varepsilon) \log 2 - 1} = e^{(1+\varepsilon) \log x - 1} \log t$$

i.e.

$$t^{(1+\varepsilon) \log 2 - 1} = \exp \{ (1+\varepsilon) \log t \}$$

... (13)

where for any $\varepsilon > 0$ with $(1+\varepsilon) \log 2 < 1$, we have for sufficiently large $x$

$$1 + \log x \log \log x \leq 2(\log x)^{(1+\varepsilon) \log 2}$$

... (14)

Hence, from (13) and (14), the number of $t$-practical numbers $n \leq x$ given in (12) becomes at most

$$\exp \{ (2c_4 \log \log x). \log(t + 1) \} \cdot \exp \{ (1+\varepsilon) \log 2 - \log t \}$$

... (15)

Thus (15) becomes at most

$$\exp \{ (c_5 \log \log x). (\log t) \log(x)^{(1+\varepsilon) \log 2} - \log t \}$$

and

$$\exp \{ (c_5 \log \log x). (\log t) \log(x)^{(1+\varepsilon) \log 2} - 1 \}$$

... (16)

where $c_5 > 0$ is a constant. Since $(1+\varepsilon) \log 2 < 1$, then, for large $x$,

$$\log(x)^{(1+\varepsilon) \log 2} \leq \log(x)^{1-\delta}$$

with $\delta > 0$ is a constant chosen such that $0 < \delta < 1 - (1+\varepsilon) \log 2$.

Thus, the number of $t$-practical numbers $n \leq x$ given in (16) is at most

$$O[\exp \{ \frac{c_5 \log t}{(\log x)^{\delta}} \}]$$

... (17)

for $t$ in the range of $1 \leq t \leq \exp \{ (\log x)^{\delta} \}$, where $0 < \delta_1 < \delta$ or (17) is written as

$$O[\exp \{ \frac{(\log x)^{\eta}}{(\log x)^{\eta}} \}]$$

and $\eta = \delta - \delta_1$. Since $\eta > 0$, then

$$O \left( x / (\log x)^{\frac{\eta}{2}} \right)$$

which is the bound for the $t$-practical numbers $n \leq x$ with $0 < \varepsilon < \left( \frac{1}{\log 2} - 1 \right)$. By putting $\beta = \frac{x^2}{2}$, then the number of $t$-practical numbers $n \leq x$ is

$$O \left( x / (\log x)^{\beta} \right)$$

Provided that $0 < \beta < \left( \frac{1}{2} \left( \frac{1}{\log 2} - 1 \right) \right)^2$.

This ends the proof and Theorem (3.2) will cover a wide range of $t$-practical numbers $n$ under the same bound given by Hausman and Shapiro [4], where Theorem (3.2) implies Theorem (3.1) [4] when $t = 1$. 

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References