Jordan Triple Higher \((\sigma, \tau)\)-Homomorphisms on Prime Rings

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Abstract

In this paper, the concept of Jordan triple higher \((\sigma, \tau)\)-homomorphisms on prime rings is introduced. A result of Herstein is extended on this concept from the ring \(R\) into the prime ring \(R'\). We prove that every Jordan triple higher \((\sigma, \tau)\)-homomorphism of ring \(R\) into prime ring \(R'\) is either triple higher \((\sigma, \tau)\)-homomorphism or triple higher \((\sigma, \tau)\)-anti-homomorphism of \(R\) into \(R'\).

Keywords: Jordan homomorphisms, triple homomorphism, Jordan triple higher \((\sigma, \tau)\)-homomorphism.

Introduction


Throughout this paper, \(R\) is a ring with the center \(Z(R)\) prime if \(aRb = (0)\) implies \(a = 0\) or \(b = 0\) with \(a, b \in R\), and is semiprime if \(aRa = (0)\) implies \(a = 0\). \(R\) is n-torsion free if \(na = 0; a \in R\), then \(a = 0\).

In this paper, we extend the result of Herstein to triple higher \((\sigma, \tau)\)-homomorphism and Jordan triple higher \((\sigma, \tau)\)-homomorphism. We show that every Jordan triple higher \((\sigma, \tau)\)-homomorphism, from prime ring \(R\) into prime ring \(R'\), is triple higher \((\sigma, \tau)\)-homomorphism or triple higher \((\sigma, \tau)\)-anti-homomorphism.

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2. Preliminaries

We begin by the following definition.

Definition 2.1. [3, 4, 5]

An additive mapping $\theta$ of a ring $R$ into a ring $R'$ is called,

(a) a homomorphism if $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in R$,
(b) anti-homomorphism if $\theta(ab) = \theta(b)\theta(a)$ for all $a, b \in R$,
(c) a Jordan homomorphism if $\theta(ab + ba) = \theta(a)\theta(b) + \theta(b)\theta(a)$ for all $a, b \in R$ and
(d) a Jordan triple homomorphism if $\theta(aba) = \theta(a)\theta(b)\theta(a)$ for all $a, b \in R$.

Obviously, every homomorphism or anti-homomorphism is a Jordan homomorphism and every Jordan homomorphism is Jordan triple homomorphism but the converse needs not to be true in general.

Definition 2.2. [6]

Let $\mathbb{N}$ be the set of natural numbers. A family of additive mappings $\theta = (\phi_i)_{i \in \mathbb{N}}$ of $R$ into $R'$ is called

(a) a higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,
$$\phi_n(ab) = \sum_{i=1}^{n} \phi_i(a) \phi_i(b),$$

(b) a higher anti-homomorphism if for all $n \in \mathbb{N}, a, b \in R$,
$$\phi_n(ab) = \sum_{i=1}^{n} \phi_i(b) \phi_i(a),$$

(c) a Jordan higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,
$$\phi_n(ab + ba) = \sum_{i=1}^{n} \phi_i(a) \phi_i(b) + \phi_i(b)\phi_i(a)$$

(d) a triple higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,
$$\phi_n(abc) = \sum_{i=1}^{n} \phi_i(a) \phi_i(b)\phi_i(c),$$

(e) a Jordan triple higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,
$$\phi_n(aba) = \sum_{i=1}^{n} \phi_i(a) \phi_i(b)\phi_i(a)$$

Definition 2.3. [7]

Let $\mathbb{N}$ be the set of natural numbers. A family of additive mappings $\theta = (\phi_i)_{i \in \mathbb{N}}$ of $R$ into $R'$ and $\sigma, \tau$ as two homomorphisms of $R$ is said to be

(a) a $(\sigma, \tau)$ —higher homomorphism if for each $n \in \mathbb{N}$ and for all $a, b \in R$,
$$\phi_n(ab) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b))$$

(b) a $(\sigma, \tau)$ —higher anti-homomorphism if for each $n \in \mathbb{N}$ and for all $a, b \in R$,
$$\phi_n(ab) = \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\tau^i(a))$$

(c) a Jordan $(\sigma, \tau)$ —higher homomorphism if for each $n \in \mathbb{N}$ and for all $a, b \in R$,
$$\phi_n(ab + ba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) + \phi_i(\sigma^i(b))\phi_i(\tau^i(a))$$

(d) a Jordan triple $(\sigma, \tau)$ —higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,
$$\phi_n(aba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a))$$
Definition 2.4.

Let \( \mathbb{N} \) be the set of natural numbers. A family of additive mappings \( \theta = (\phi_i)_{i \in \mathbb{N}} \) of \( R \) into \( R' \) and \( \sigma, \tau \) as two homomorphisms of \( R \) is said to be

(a) a triple \((\sigma, \tau)\)-higher homomorphism if for all \( n \in \mathbb{N}, a, b \in R \),

\[
\phi_n(abc) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))
\]

(b) a triple \((\sigma, \tau)\)-higher anti-homomorphism if for all \( n \in \mathbb{N}, a, b \in R \),

\[
\phi_n(abc) = \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a)).
\]

Now, we give an example of triple \((\sigma, \tau)\)-higher homomorphism and Jordan triple \((\sigma, \tau)\)-higher homomorphism.

Example 2.5:

Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a triple \((\sigma, \tau)\)-higher homomorphism from \( R \) into \( R' \). Then for each \( n \in \mathbb{N} \) and for all \( a, b, c \in R \), we have:

\[
\phi_n(abc) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))
\]

Let \( T = R \times R \times R \) and \( T' = R' \times R' \times R' \). Then \( T \) and \( T' \) are rings. We define \( \theta' = (\phi'_i)_{i \in \mathbb{N}} \) to be a family of mappings from \( T \) to \( T' \) by:

\[
\phi'_n((a, b, c)) = (\phi_n(a), \phi_n(b), \phi_n(c))
\]

for all \((a, b, c) \in T\).

Then \( \phi \) is a triple \((\sigma, \tau)\)-higher homomorphism.

Let \( S \) be the subset \( \{(a, a, a) : a \in R \} \) of \( T \) and \( S' \) be the subset \( \{(b, b, b) : b \in R' \} \) of \( T' \). Then \( S \) and \( S' \) are rings and the family of mappings \( \theta' = (\phi'_i)_{i \in \mathbb{N}} \) from \( S \) to \( S' \) is defined in terms of the Jordan \((\sigma, \tau)\)-higher homomorphism by

\[
\phi'_n((a, a, a)) = (\phi_n(a), \phi_n(b), \phi_n(c))
\]

for all \((a, a, a) \in S\).

Then \( \phi \) is a Jordan triple \((\sigma, \tau)\)-higher homomorphism from \( S \) to \( S' \).

Obviously, every triple \((\sigma, \tau)\)-higher homomorphism or triple \((\sigma, \tau)\)-higher anti-homomorphism is a Jordan triple \((\sigma, \tau)\)-higher homomorphism but the converse needs not to be true in general.

In an earlier work[6], the author provided an example of Jordan higher homomorphism but not higher homomorphism on a ring. We extend it to triple \((\sigma, \tau)\)-higher homomorphism on rings as follows.

Example 2.6.

Suppose that \( S \) is a ring with non-trivial involution * on \( R = S \oplus S \oplus S \), \( a \in S \) such that \( a \in Z(S) \) and \( s_1as_2 = 0 \), for all \( s_1, s_2 \in R \). Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a family of mappings of \( R \) into itself defined, for each \( n \in \mathbb{N} \) and \((s, t, s) \in R \), by:

\[
\phi_n(s, t, s) = \begin{cases} (2 - n)a\sigma^i(s), (n - 1)\sigma^i\tau^{n-i}(t), (2 - n)a\sigma^i(s) & \text{for } n = 1, 2 \\ 0 & \text{for } n \geq 3 \end{cases}
\]

Therefore, it is clear that \( \phi \) is a Jordan triple \((\sigma, \tau)\)-higher homomorphism but not a triple \((\sigma, \tau)\)-higher homomorphism.

Now, we will give the following lemmas which are used in the proofs of the main results.

Lemma 2.7: [5]

Let \( R \) be a 2-torsion free semiprime ring. If \( x, y \in R \) such that \( xry + yr(x) = 0 \), for all \( r \in R \), then \( xry = yr(x) = 0 \).

Lemma 2.8:

Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan triple \((\sigma, \tau)\)-higher homomorphism of \( R \) into \( R' \). Then for each \( n \in \mathbb{N} \) and for all \( a, b, c \in R \),

\[
\phi_n(abc + cba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))
\]

\[
+ \phi_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a))
\]

Proof: Since \( \phi \) is a Jordan triple \((\sigma, \tau)\)-higher homomorphism, hence
By linearizing \( a \), we get
\[
\phi_n((a + c)b(a + c)) = \sum_{i=1}^{n} \phi_i(a + c) \left( \sum_{i=1}^{n} \phi_i \left( \sigma^i(a) \right) \phi_i \left( \sigma^i \tau^{n-i}(b) \right) \phi_i \left( \tau^i(a) \right) \right) + \phi_i \left( \sigma^i(c) \right) \phi_i \left( \sigma^i \tau^{n-i}(b) \right) \phi_i \left( \tau^i(a) \right) + \phi_i \left( \sigma^i(c) \right) \phi_i \left( \sigma^i \tau^{n-i}(b) \right) \phi_i \left( \tau^i(c) \right)
\]
(1)

On the other hand:
\[
\phi_n((a + c)b(a + c)) = \phi_n(aba + abc + cba + cbc) = \phi_n(aba) + \phi_n(abc + cba) + \phi_n(cbc)
\]
\[
= \sum_{i=1}^{n} \phi_i(a) \phi_i \left( \sigma^i \tau^{n-i}(b) \right) \phi_i \left( \tau^i(a) \right) + \phi_i \left( \sigma^i(c) \right) \phi_i \left( \sigma^i \tau^{n-i}(b) \right) \phi_i \left( \tau^i(c) \right) + \phi_n(abc + cba)
\]
\[
... (2)
\]

By comparing (1) and (2), we achieve the result.

**Remark 2.9:**
Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan triple \((\sigma, \tau)\)-higher homomorphism from \( R \) into \( R' \). Then for each \( n \in \mathbb{N} \) and for all \( a, b \in R \), we will write
\[
A_n(a, b, c) = \phi_n(abc) - \sum_{i=1}^{n} \phi_i(a) \phi_i \left( \sigma^i \tau^{n-i}(b) \right) \phi_i \left( \tau^i(c) \right)
\]
\[
B_n(a, b, c) = \phi_n(abc) - \sum_{i=1}^{n} \phi_i(c) \phi_i \left( \sigma^i \tau^{n-i}(b) \right) \phi_i \left( \tau^i(a) \right)
\]

Note that \( A_n(a, b, c) = 0 \), if and only if \( \phi \) is a triple \((\sigma, \tau)\)-higher homomorphism, and \( B_n(a, b, c) = 0 \), if and only if \( \phi \) is a triple \((\sigma, \tau)\)-higher anti-homomorphism.

For the purpose of this paper, we can list the following elementary properties about the above:

1. \( A_n(a, b, c) + A_n(c, b, a) = 0 \),
2. \( B_n(a, b, c) + B_n(c, b, a) = 0 \).

**Lemma 2.10:**
If \( \theta = (\phi_i)_{i \in \mathbb{N}} \) is a Jordan triple \((\sigma, \tau)\)-higher homomorphism from a ring \( R \) into a ring \( R' \), then for all \( a, b \in R \) and \( n \in \mathbb{N} \),

i) \( A_n(a + b, c, d) = A_n(a, c, d) + A_n(b, c, d) \)
ii) \( A_n(a, b + c, d) = A_n(a, b, d) + A_n(a, c, d) \)
iii) \( A_n(a, b, c + d) = A_n(a, b, c) + A_n(a, b, d) \)

**Proof:**

i) \( A_n(a + b, c, d) = \phi_n((a + b)cd) - \sum_{i=1}^{n} \phi_i \left( \sigma^i(a + b) \right) \phi_i \left( \sigma^i \tau^{n-i}(c) \right) \phi_i \left( \tau^i(d) \right) \)
\[
= \phi_n(acd + bcd) - \sum_{i=1}^{n} \phi_i(a) \phi_i \left( \sigma^i \tau^{n-i}(c) \right) \phi_i \left( \tau^i(d) \right) - \sum_{i=1}^{n} \phi_i(b) \phi_i \left( \sigma^i \tau^{n-i}(c) \right) \phi_i \left( \tau^i(d) \right)
\]

Since \( \phi_n \) is an additive mapping for each \( n \), then
\[
= \phi_n(acd) - \sum_{i=1}^{n} \phi_i \left( \sigma^i(a) \right) \phi_i \left( \sigma^i \tau^{n-i}(c) \right) \phi_i \left( \tau^i(d) \right) + \phi_n(bcd) - \sum_{i=1}^{n} \phi_i(b) \phi_i \left( \sigma^i \tau^{n-i}(c) \right) \phi_i \left( \tau^i(d) \right) = A_n(a, c, d) + A_n(b, c, d)
\]

In a similarly way, we can prove (ii) and (iii).
3. Main Results

Lemma 3.1: If $\theta = (\theta_i)_{i \in \mathbb{N}}$ is a Jordan triple higher $(\sigma, \tau)$-homomorphism of $R$ into $R'$, then for each $n \in \mathbb{N}$ and for all $a, b, c, r \in R$, 
\[ A_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(a,b,c)) + B_n(\sigma^n(abc))\phi_n(\sigma^n(r))A_n(\tau^n(a,b,c)) = 0. \]

Proof: We proceed by the induction on $n \in \mathbb{N}$. Assume that $\theta$ is a Jordan triple higher $(\sigma, \tau)$-homomorphism and take $a, b, c, r \in R$.

If $n = 1$: Define $w = abcrca + cbara$. Then we get the required result.

We can assume that the following equation is true for all $a, b, c, r \in R, n \in \mathbb{N}$ and $m < n$:
\[ A_m(\sigma^m(a,b,c))\phi_m(\sigma^m(r))B_m(\tau^m(a,b,c)) + B_m(\sigma^m(abc))\phi_m(\sigma^m(r))A_m(\tau^m(a,b,c)) = 0. \]

Now, we have
\[ \phi_n(w) = \phi_n(a(brca)c + c(bara)c) \]
\[ = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^i(brcar))\phi_i(\tau^i(a)) + \phi_i(\sigma^i(c))\phi_i(\sigma^i\tau^i(bara))\phi_i(\tau^i(c)) \]
\[ = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b))\phi_i(\sigma^i\tau^i(c)) + \phi_i(\sigma^i(c))\phi_i(\sigma^i(c))\phi_i(\sigma^i\tau^i(c)) \]
\[ + \phi_i(\sigma^i(c))\phi_i(\sigma^i(b))\phi_i(\sigma^i\tau^i(ara))\phi_i(\tau^i(b))\phi_i(\tau^i(c)) \]
\[ = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b))\left( \sum_{j=1}^{i} \phi_j(\sigma^j\tau^j(ara))\phi_j(\tau^j(b))\phi_j(\tau^j(c)) \right) \]
\[ + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i(b))\left( \sum_{j=1}^{i} \phi_j(\sigma^j\tau^j(ara))\phi_j(\tau^j(b))\phi_j(\tau^j(c)) \right) \]
\[ = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b))\phi_i(\sigma^i\tau^i(c)) + \phi_i(\sigma^i(c))\phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^i(c)) \]
\[ + \phi_i(\sigma^i(c))\phi_i(\sigma^i(b))\phi_i(\sigma^i\tau^i(ara))\phi_i(\tau^i(b))\phi_i(\tau^i(c)) \]
\[ = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b))\phi_i(\sigma^i\tau^i(c)) + \phi_i(\sigma^i(c))\phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^i(c)) \]
\[ + \phi_i(\sigma^i(c))\phi_i(\sigma^i(b))\phi_i(\sigma^i\tau^i(ara))\phi_i(\tau^i(b))\phi_i(\tau^i(c)) \]
\[ = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b))\phi_i(\sigma^i\tau^i(r))\sum_{j=1}^{i} \phi_j(\tau^j\sigma^j\tau^j(a))\phi_j(\tau^j(b))\phi_j(\tau^j(c)) \]
\[ + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i(b))\phi_i(\sigma^i\tau^i(r))\sum_{j=1}^{i} \phi_j(\tau^j\sigma^j\tau^j(a))\phi_j(\tau^j(b))\phi_j(\tau^j(c)) \]
\[ = \phi_n(\sigma^n(a))\phi_n(\sigma^n(b))\phi_n(\sigma^n\sigma^n(c))\phi_n(\sigma^n\sigma^n(r))\sum_{j=1}^{i} \phi_j(\tau^j\sigma^j\tau^j(a))\phi_j(\tau^j(b))\phi_j(\tau^j(c)) \]
\[ + \phi_n(\sigma^n(a))\phi_n(\sigma^n(b))\phi_n(\sigma^n\sigma^n(c))\phi_n(\sigma^n\sigma^n(r))\sum_{j=1}^{i} \phi_j(\tau^j\sigma^j\tau^j(a))\phi_j(\tau^j(b))\phi_j(\tau^j(c)) \]
\[ + \phi_n(\sigma^n(c))\phi_n(\sigma^n(b))\phi_n(\sigma^n\sigma^n(a))\phi_n(\sigma^n\sigma^n(r))\sum_{j=1}^{i} \phi_j(\tau^j\sigma^j\tau^j(a))\phi_j(\tau^j(b))\phi_j(\tau^j(c)) \]
\[ + \phi_n(\sigma^n(c))\phi_n(\sigma^n(b))\phi_n(\sigma^n\sigma^n(a))\phi_n(\sigma^n\sigma^n(r))\sum_{j=1}^{i} \phi_j(\tau^j\sigma^j\tau^j(a))\phi_j(\tau^j(b))\phi_j(\tau^j(c)) \]
\[ \begin{align*}
&= \phi_n(w) = \phi_n((abc)r(cba) + (cba)r(abc)) \quad \ldots (3)
\end{align*} \]
\[= \sum_{i=1}^{n} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(cb\alpha)) + f_i(\sigma^i(cb\alpha)) \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(abc))\]

Since \(\theta^i\) is a Jordan triple higher \((\sigma,\tau)\)-homomorphism, then

\[= \sum_{i=1}^{n} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{n-i}(r)) \left(\sum_{j=1}^{i} \phi_j(\tau^j(c)) \phi_j(\tau^j(b)) \phi_j(\tau^j(a)) \right)\]

\[+ \phi_j(\tau^j(a)) \phi_j(\tau^j(b)) \phi_j(\tau^j(c)) - \phi_j(\tau^j(abc)) \]

\[+ \sum_{i=1}^{n} \sum_{j=1}^{i} \phi_j(\sigma^j\sigma^i(c)) \phi_j(\sigma^j\tau^{i-j}(\sigma^i(b))) \phi_j(\tau^i(\sigma^j(a))) \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(abc))\]

\[= \sum_{i=1}^{n} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{n-i}(r)) \sum_{j=1}^{i} \phi_j(\tau^j(c)) \phi_j(\tau^j(b)) \phi_j(\tau^j(a))\]

\[+ \sum_{i=1}^{n} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{n-i}(r)) \sum_{j=1}^{i} \phi_j(\tau^j(a)) \phi_j(\tau^j(b)) \phi_j(\tau^j(c))\]

\[- \sum_{i=1}^{n} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(abc))\]

\[+ \sum_{i=1}^{n} \sum_{j=1}^{i} \phi_j(\sigma^j\sigma^i(c)) \phi_j(\sigma^j\tau^{i-j}(\sigma^i(b))) \phi_j(\tau^i(\sigma^j(a))) \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(abc))\]

\[+ \sum_{i=1}^{n} \sum_{j=1}^{i} \phi_j(\sigma^j\sigma^i(a)) \phi_j(\sigma^j\tau^{i-j}(\sigma^i(b))) \phi_j(\tau^i(\sigma^j(c))) \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(abc))\]

\[- \sum_{i=1}^{n} \sum_{j=1}^{i} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(abc))\]

\[= - \sum_{i=1}^{n} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{n-i}(r)) \left(\sum_{j=1}^{i} \phi_i(\tau^i(abc)) - \phi_j(\tau^j(c)) \phi_j(\tau^j(b)) \phi_j(\tau^j(a)) \right)\]

\[- \sum_{i=1}^{n} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{n-i}(r)) \left(\sum_{j=1}^{i} \phi_i(\tau^i(abc)) - \phi_j(\tau^j(a)) \phi_j(\tau^j(b)) \phi_j(\tau^j(c)) \right)\]

\[+ \sum_{i=1}^{n} \phi_i(\sigma^i\sigma^i(c)) \phi_i(\sigma^i\tau^{n-i}(\sigma^i(b))) \phi_i(\tau^i(\sigma^i(a))) \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(abc))\]

\[+ \sum_{i=1}^{n} \phi_i(\sigma^i\sigma^i(a)) \phi_i(\sigma^i\tau^{n-i}(\sigma^i(b))) \phi_i(\tau^i(\sigma^i(c))) \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(abc))\]
\[\begin{align*}
&= - \sum_{i=1}^{n} \phi_i \left( \sigma^{i}(abc) \right) \phi_i \left( \sigma^{i} \tau^{n-i}(r) \right) B_i \left( \tau^{i}(a), \tau^{i}(b), \tau^{i}(c) \right) \\
&\quad - \sum_{i=1}^{n} \phi_i \left( \sigma^{i}(abc) \right) \phi_i \left( \sigma^{i} \tau^{n-i}(r) \right) A_i \left( \tau^{i}(a), \tau^{i}(b), \tau^{i}(c) \right) \\
&\quad + \sum_{i=1}^{n} \phi_i \left( \sigma^{i} \sigma^{i}(c) \right) \phi_i \left( \sigma^{i} \tau^{n-i} \sigma^{i}(b) \right) \phi_i \left( \tau^{i}(a) \right) \phi_i \left( \sigma^{i} \tau^{n-i}(r) \right) \phi_i \left( \tau^{i}(abc) \right) \\
&\quad + \sum_{i=1}^{n} \phi_i \left( \sigma^{i} \sigma^{i}(a) \right) \phi_i \left( \sigma^{i} \tau^{n-i} \sigma^{i}(b) \right) \phi_i \left( \tau^{i}(c) \right) \phi_i \left( \sigma^{i} \tau^{n-i}(r) \right) \phi_i \left( \tau^{i}(abc) \right) \\
&= - \phi_n \left( \sigma^{n}(abc) \right) \phi_n \left( \sigma^{n}(r) \right) B_n \left( \tau^{n}(a, b, c) \right) - \sum_{i=1}^{n-1} \phi_i \left( \sigma^{i}(abc) \right) \phi_i \left( \sigma^{i} \tau^{n-i}(r) \right) A_i \left( \tau^{i}(a, b, c) \right)
\end{align*}\]

From equation (3) and (4), we get

\[\begin{align*}
0 &= - \phi_n \left( \sigma^{n}(abc) \right) \phi_n \left( \sigma^{n}(r) \right) B_n \left( \tau^{n}(a, b, c) \right) - \phi_n \left( \sigma^{n}(abc) \right) \phi_n \left( \sigma^{n}(r) \right) A_n \left( \tau^{n}(a, b, c) \right) \\
&\quad + \sum_{i=1}^{n-1} \phi_i \left( \sigma^{i} \sigma^{i}(c) \right) \phi_i \left( \sigma^{i} \tau^{n-i} \sigma^{i}(b) \right) \phi_i \left( \tau^{i}(a) \right) \phi_i \left( \sigma^{i} \tau^{n-i}(r) \right) \phi_i \left( \tau^{i}(abc) \right) \\
&\quad \quad - \phi_i \left( \tau^{i}(a) \right) \phi_i \left( \tau^{i}(b) \right) \phi_i \left( \tau^{i}(c) \right) \phi_i \left( \tau^{i}(abc) \right) \\
&\quad + \sum_{i=1}^{n-1} \phi_i \left( \sigma^{i} \sigma^{i}(a) \right) \phi_i \left( \sigma^{i} \tau^{n-i} \sigma^{i}(b) \right) \phi_i \left( \tau^{i}(c) \right) \phi_i \left( \sigma^{i} \tau^{n-i}(r) \right) \phi_i \left( \tau^{i}(abc) \right) \\
&\quad \quad - \phi_i \left( \tau^{i}(c) \right) \phi_i \left( \tau^{i}(b) \right) \phi_i \left( \tau^{i}(a) \right) \phi_i \left( \tau^{i}(abc) \right) \\
&\quad + \phi_n \left( \sigma^{n}(c) \right) \phi_n \left( \sigma^{n}(b) \right) \phi_n \left( \sigma^{n}(a) \right) \phi_n \left( \sigma^{n}(r) \right) \phi_n \left( \tau^{n}(abc) \right) \\
&\quad - \sum_{i=1}^{n-1} \phi_i \left( \tau^{i}(a) \right) \phi_i \left( \tau^{i}(b) \right) \phi_i \left( \tau^{i}(c) \right) \\
&\quad + \phi_n \left( \sigma^{n}(a) \right) \phi_n \left( \sigma^{n}(b) \right) \phi_n \left( \sigma^{n}(c) \right) \phi_n \left( \sigma^{n}(r) \right) \phi_n \left( \tau^{n}(abc) \right) \\
&\quad - \sum_{i=1}^{n-1} \phi_i \left( \tau^{i}(c) \right) \phi_i \left( \tau^{i}(b) \right) \phi_i \left( \tau^{i}(a) \right)
\end{align*}\]
\[
- \sum_{i=1}^{n-1} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{i-1}(r)) A_i(\tau^i(a, b, c)) \\
- \sum_{i=1}^{n-1} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{i-1}(r)) B_i(\tau^i(a, b, c)) \\
= -\phi_n(\sigma^n(abc)) \phi_n(\sigma^n(a)) \phi_n(\sigma^n(b)) \phi_n(\sigma^n(c)) A_n(\tau^n(a, b, c)) \\
+ \phi_n(\sigma^n(c)) \phi_n(\sigma^n(b)) \phi_n(\sigma^n(a)) \phi_n(\sigma^n(r)) A_n(\tau^n(a, b, c)) \\
+ \sum_{i=1}^{n-1} \phi_i(\sigma^i\tau^{i-1}(r)) A_i(\tau^i(a, b, c)) \\
+ \sum_{i=1}^{n-1} \phi_i(\sigma^i\tau^{i-1}(r)) B_i(\tau^i(a, b, c)) \\
- \sum_{i=1}^{n-1} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{i-1}(r)) A_i(\tau^i(a, b, c)) \\
- \sum_{i=1}^{n-1} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{i-1}(r)) B_i(\tau^i(a, b, c)) \\
= -\phi_n(\sigma^n(abc)) \phi_n(\sigma^n(a)) \phi_n(\sigma^n(b)) \phi_n(\sigma^n(c)) A_n(\tau^n(a, b, c)) \\
- \phi_n(\sigma^n(a)) \phi_n(\sigma^n(b)) \phi_n(\sigma^n(c)) \phi_n(\sigma^n(r)) A_n(\tau^n(a, b, c)) \\
- \sum_{i=1}^{n-1} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{i-1}(r)) A_i(\tau^i(a, b, c)) \\
- \sum_{i=1}^{n-1} \phi_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{i-1}(r)) B_i(\tau^i(a, b, c)) \\
= -\phi_n(\sigma^n(a, b, c)) \phi_n(\sigma^n(r)) B_n(\tau^n(a, b, c)) \\
- \sum_{i=1}^{n-1} B_i(\sigma^i(abc)) \phi_i(\sigma^i\tau^{i-1}(r)) A_n(\tau^n(a, b, c)) \\
- \sum_{i=1}^{n-1} A_n(\sigma(a, b, c)) \phi_i(\sigma^i\tau^{i-1}(r)) B_n(\tau^n(a, b, c)) \\
Hence, we have \\
A_n(\sigma^a(a, b, c)) \phi_n(\sigma^n(r)) B_n(\tau^n(a, b, c)) + B_n(\sigma^a(a, b, c)) \phi_n(\sigma^n(r)) A_n(\tau^n(a, b, c)) = 0.

Lemma 3.2: 
Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan triple higher \((\sigma, \tau)\)-homomorphism of \( R \) into \( R' \), then for each \( n \in \mathbb{N} \) and for all \( a, b, c, r, \in R \), 
\[ A_n(\sigma^n(a, b, c)) \phi_n(\sigma^n(r)) B_n(\tau^n(a, b, c)) = B_n(\sigma^n(a, b, c)) \phi_n(\sigma^n(r)) A_n(\tau^n(a, b, c)) = 0. \]

Proof.
By Lemma 3.1 and Lemma 2.7, we achieve the result.

Theorem 3.3: 
Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan triple higher \((\sigma, \tau)\)-homomorphism of ring \( R \) into prime ring \( R' \). Then for each \( n \in \mathbb{N} \) and for all \( a, b, c, r, x, y, z \in R \), 
\[ A_n(\sigma^n(a, b, c)) \phi_n(\sigma^n(r)) B_n(\tau^n(a, b, c)) = B_n(\sigma^n(a, b, c)) \phi_n(\sigma^n(r)) A_n(\tau^n(a, b, c)) = 0. \]

Proof.
By replacing \( a + x \) by \( a \) in Lemma 3.2, we get
Hence
\[ A_n\left(\sigma^n(a + x, b, c)\right) \phi_n \left(\sigma^n(r)\right) B_n \left(\tau^n(a + x, b, c)\right) = 0 \]

By Lemma 3.2, we obtain
\[ A_n\left(\tau^n(a, b, c)\right) \phi_n \left(\sigma^n(r)\right) B_n \left(\tau^n(a, b, c)\right) = 0 \]

Since \( R' \) is prime, we obtain
\[ A_n\left(\sigma^n(a, b, c)\right) \phi_n \left(\sigma^n(r)\right) B_n \left(\tau^n(a, b, c)\right) = 0. \]... (5)

By replacing \( b + y \) for \( b \) in equation (5), we get
\[ A_n\left(\sigma^n(a, b + y, c)\right) \phi_n \left(\sigma^n(r)\right) B_n \left(\tau^n(a, b + y, c)\right) = 0 \]

By replacing \( c + z \) for \( c \) in equation (6), we get
\[ A_n\left(\sigma^n(a, b + c + z)\right) \phi_n \left(\sigma^n(r)\right) B_n \left(\tau^n(a, b + c + z)\right) = 0 \]

Since \( R' \) is prime, we obtain
\[ A_n\left(\sigma^n(a, b, c)\right) \phi_n \left(\sigma^n(r)\right) B_n \left(\tau^n(a, b, c)\right) = 0. \]... (6)

In the following theorem we give the conditions which make the Jordan triple higher \((\sigma, r)\)-homomorphism is either triple higher \((\sigma, r)\)-homomorphism or triple higher \((\sigma, r)\)-anti-homomorphism.

**Theorem 3.4:**
Every Jordan triple higher \((\sigma, r)\)-homomorphism of ring \( R \) into prime ring \( R' \) is either triple higher \((\sigma, r)\)-homomorphism or triple higher \((\sigma, r)\)-anti-homomorphism.

**Proof.**
Let \( \theta \) be a Jordan triple higher \((\sigma, r)\)-homomorphism. Then by Theorem 3.3, we have
\[ A_n\left(\sigma^n(a, b, c)\right) \phi_n \left(\sigma^n(r)\right) B_n \left(\tau^n(a, b, c)\right) = 0 \]

Since \( R' \) is prime, therefore either \( A_n\left(\sigma^n(a, b, c)\right) = 0 \) or \( B_n\left(\tau^n(a, b, c)\right) = 0 \), for each \( n \in \mathbb{N} \) and for all \( a, b, c, x, y, z \in R \).

If \( B_n\left(\tau^n(x, y, z)\right) = 0 \), then by Remark 2.9, we obtain \( \theta \) is triple higher \((\sigma, r)\)-anti-homomorphism.

But if \( A_n\left(\sigma^n(a, b, c)\right) = 0 \), then by Remark 2.9, we obtain \( \theta \) is triple higher \((\sigma, r)\)-homomorphism.
Proposition 3.5:
Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan triple higher \((\sigma, \tau)\)-homomorphism from prime ring \( R \) into prime ring \( R' \), then \( \theta \) is higher \((\sigma, \tau)\)-homomorphism.

**Proof:**
Since \( \theta \) is a Jordan triple higher \((\sigma, \tau)\)-homomorphism, then for all \( a, r \in R \) and \( n \in \mathbb{N} \), we have
\[
\phi_n(ar) = \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \phi_i(\sigma^i \tau^{n-i}(r)) \phi_i(\tau^i(a))
\]

By replacing \( a \) by \( ab \), we get
\[
\phi_n((ab)r(ab)) = \sum_{i=1}^{n} \phi_i(\sigma^i(ab)) \phi_i(\sigma^i \tau^{n-i}(r)) \phi_i(\tau^i(ab))
\]
\[
= \phi_n(\sigma^n(ab))rab + ab \sum_{i=1}^{n} \phi_i(\sigma^i \tau^{n-i}(r)) \phi_i(\tau^i(ab))
\]
\[\text{... (7)}\]

On the other hand, we get
\[
\phi_n((ab)r(ab)) = \sum_{i=1}^{n} \phi_i(\sigma^i(ab)) \phi_i(\sigma^i \tau^{n-i}(r)) \phi_i(\tau^i(ab))
\]
\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \phi_i(\sigma^i(b)) rab + ab \sum_{i=1}^{n} \phi_i(\sigma^i \tau^{n-i}(r)) \phi_i(\tau^i(ab))
\]
\[\text{... (8)}\]

By comparing (7) and (8), we get
\[
\left( \phi_n(\sigma^n(ab)) - \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \phi_i(\sigma^i(b)) \right) rab = 0.
\]

Since \( R \) is prime and \( ab \neq 0 \), we get
\[
\phi_n(ab) = \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \phi_i(\tau^i(b))
\]

Hence \( \theta \) is a higher \((\sigma, \tau)\)-homomorphism.

**References**