On Soft $P_c$-Connected Spaces

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Abstract
In this paper, we define the concept of soft $p_c$-connected sets and soft $p_c$-connected spaces by using the notion of soft $p_c$-open sets in soft topological spaces. Several properties of these concepts are investigated.

Keywords: (soft $p_c$-open set, soft $p_c$-separated sets, soft $p_c$-connected, soft $p_c$-disconnected)

1 Introduction
In topology, connectedness is used to refer to various properties meaning in some sense, (all in one piece). When a mathematical object has such a property, we say that it is connected; otherwise, it is disconnected. Connectivity occupies very important place in topology. Many authors have presented different kinds of connectivity in general, including fuzzy settings and intuitionistic fuzzy settings such as $P$-connectedness and semi-pre connectedness in intuitionistic fuzzy topological spaces. After the foundation of the soft set theory by D. Molodtsov [1], many researchers studied soft topological structures. In 2012, Peyhan et al. [2] introduced soft connectedness in soft topological spaces. In 2013, Lin [3] continued the study of soft connectedness. In 2015, Husain [4] provided more characterizations of soft connectedness in soft topological spaces. In the present paper, we introduce another type of soft separation and soft connectedness in soft topological spaces by using the concept of soft $p_c$-open and soft $p_c$-closed sets, called soft $p_c$-separation and soft $p_c$-connected space. We also describe the essential properties of these concepts along with the relations between these types and other existing types.

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Throughout this paper, $X$ will be a nonempty initial universal set and $A$ will be a set of parameters. A pair $(F, A)$ is called a soft set over $X$, where $F$ is a function $F: A \rightarrow P(X)$. The collection of soft sets $(F, E)$ over a universal set $X$ with the parameter set $A$ is denoted by $SP(X)_A$. Any logical operation ($\land$) on soft sets in soft topological spaces is denoted by the usual set theoretical operations with the symbol ($\wedge$).

### 2 Preliminaries

This section contains the main definitions and results in soft topological spaces which are needed in other sections. All these definitions can be found in several articles concerning soft set theory and soft topological spaces [5-9].

**Definition 2.1** [5] A soft set $(F, A)$ over $X$ is said to be empty soft set denoted by $\tilde{\Phi}$ if for all $e \in A$, $F(e) = \phi$ and $(F, A)$ over $X$ is said to be absolute soft set denoted by $\tilde{X}$ if for all $e \in A$, $F(e) = X$.

**Definition 2.2** [5] The complement of a soft set $(F, A)$ is denoted by $(F, A)^c$ or $\tilde{X} \setminus (F, A)$ and is defined by $(F, A)^c = (F^c, A)$ where $F^c: A \rightarrow P(X)$ is a function given by $F^c(e) = X \setminus F(e)$, for all $e \in A$.

It is clear that $((F, A)^c)^c = (F, A)$, $\tilde{\Phi}^c = \tilde{X}$ and $\tilde{X}^c = \tilde{\Phi}$.

**Definition 2.3** [5] For two soft sets $(F, A)$ and $(G, B)$ over a common universe $X$, we say that $(F, A)$ is a soft subset of $(G, B)$, if
1. $A \subseteq B$ and
2. for all $e \in A$, $F(e) \subseteq G(e)$

We write $(F, A) \subseteq (G, B)$.

**Definition 2.4** [5] The union of two soft sets of $(F, A)$ and $(G, B)$ over the common universe $X$ is the soft set $(H, C) = (F, A) \cup (G, B)$, where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \cup G(e) & \text{if } e \in A \cap B \end{cases}$$

In particular, $(F, A) \cup (G, A) = F(e) \cup G(e)$ for all $e \in A$.

**Definition 2.5** [5] The intersection $(H, C)$ of two soft sets $(F, A)$ and $(G, B)$ over a common universe $X$, denoted $(F, A) \cap (G, B)$, is defined as $C = A \cap B$, and $H(e) = F(e) \cap G(e)$ for all $e \in C$.

In particular, $(F, A) \cap (G, A) = F(e) \cap G(e)$ for all $e \in A$.

**Definition 2.6** [7] The soft set $(F, A)$ is called a soft point, denoted by $(x_e, A)$ or $x_e$, if for the element $e \in A$, $F(e) = \{x\}$ and $F(e') = \phi$ for all $e' \in E \setminus \{e\}$.

We say that $x_e \in (G, A)$ if $x \in G(e)$.

Two soft points $x_e$ and $y_{e'}$ are distinct if either $x \neq y$ or $e \neq e'$.

**Definition 2.7** [10] Let $\tilde{\tau}$ be a collection of soft sets over a universe $X$ with a fixed set $E$ of parameters. Then $\tilde{\tau} \subseteq SP(X)_A$ is called a soft topology, if
1. $\tilde{\Phi}$ and $\tilde{X}$ belongs to $\tilde{\tau}$
2. The union of any number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$
3. The intersection of any two soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

The triplet $(X, \tilde{\tau}, A)$ is called a soft topological space over $X$. The members of $\tilde{\tau}$ are called soft open sets in $\tilde{X}$ while the complements of them are called soft closed sets in $\tilde{X}$, and they are denoted by $SO(\tilde{X})$ and $SC(\tilde{X})$, respectively. Soft interior and soft closure are denoted by $\tilde{\text{int}}$ and $\tilde{\text{cl}}$, respectively.

**Definition 2.8** [10] Let $(X, \tilde{\tau}, A)$ be a soft topological space and let $(G, A)$ be a soft set. Then,
1. The soft closure of $(G, A)$ is the soft set $\tilde{\text{cl}}(G, A) = \tilde{\cap} \{(K, B) \in SC(\tilde{X}); (G, A) \subseteq (K, B)\}$
2. The soft interior of $(G, A)$ is the soft set $\tilde{\text{int}}(G, A) = \tilde{\cup} \{(H, B) \in SO(\tilde{X}); (H, B) \subseteq (G, A)\}$.

**Definition 2.9** [11] Let $(X, \tilde{\tau}, A)$ be a soft topological space, $(G, A)$ is a soft set over $\tilde{X}$ and $x_e \notin \tilde{X}$. Then $(G, A)$ is said to be a soft neighborhood of $x_e$, if there exists a soft open set $(H, A)$ such that $x_e \in (H, A) \subseteq ((G, A))$.  

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Proposition 2.10 [10] Let \((Y, \bar{\tau}, \mathcal{A})\) be a soft subspace of a soft topological space \((X, \bar{\tau}, \mathcal{A})\) and \((F, A) \in \mathcal{SP}(X)_E\). Then:

1. If \((F, A)\) is a soft open set in \(Y\) and \(Y \in \bar{\tau}\), then \((F, A) \in \bar{\tau}\).
2. \((F, A)\) is a soft open set in \(Y\) if and only if \((F, A) = \bar{Y} \cap (G, A)\) for some \((G, A) \in \bar{\tau}\).
3. \((F, A)\) is a soft closed set in \(Y\) if and only if \((F, A) = \overline{Y} \cap (H, A)\) for some soft closed \((H, A)\) in \(X\).

Definition 2.11 [12] A soft subset \((F, A)\) of a soft space \(X\) is said to be soft pre-open if \((F, A) \subseteq \overline{\text{Int}\mathcal{Scl}}(F, A)\). The complement of soft pre-open set is said to be soft pre-closed. The family of soft pre-open (soft pre-closed) set is denoted by \(\mathcal{SPC}(X)\) and \(\mathcal{SPC}(X)\) respectively.

Lemma 2.1 [12] Arbitrary union of soft pre-open sets is a soft pre-open set.

Definition 2.12 [13] Let \((X, \bar{\tau}, \mathcal{A})\) be a soft topological space and let \((G, A)\) be a soft set. Then

1. The soft pre-closure of \((G, A)\) is the soft set

\[\bar{\text{spc}}(G, A) = \overline{\{ (K, B) \in \mathcal{SP}(X); (G, A) \subseteq (K, B) \}\}\]

2. The soft pre-interior of \((G, A)\) is the soft set

\[\text{spint}(G, A) = \overline{\{ (H, B) \in \mathcal{SP}(X); (H, B) \subseteq (G, A) \}\}\].

In [14], Bayramov and Aras defined a soft \(T_1\)-space as:

Definition 2.13 [14] A soft topological space \((X, \bar{\tau}, \mathcal{A})\) is said to be soft \(T_1\), if for each pair of distinct soft points \(x_{e_1}, y_{e_2} \in \mathcal{SP}(X)\), there exist two soft open sets \((F, A)\) and \((G, A)\) such that \(x_{e_1} \in (F, A)\) but \(y_{e_2} \notin (F, A)\) and \(y_{e_2} \in (G, A)\) but \(x_{e_1} \notin (G, A)\).

Proposition 2.14 [14] A soft topological space \((X, \bar{\tau}, \mathcal{A})\) is soft \(T_1\) if and only if each soft point is soft closed.

Definition 2.15 [15] A soft pre-open set \((F, A)\) in a soft topological space \((X, \bar{\tau}, \mathcal{A})\) is called soft \(p_e\)-open if, for each \(x_{e} \in (F, A)\), there exists a soft closed set \((K, A)\) such that \(x_{e} \in (K, A) \subseteq (F, A)\). The soft complement of each soft \(p_e\)-open set is called soft \(p_e\)-closed set.

The family of all soft \(p_e\)-open (resp., soft \(p_e\)-closed) sets in a soft topological space \((X, \bar{\tau}, \mathcal{A})\) is denoted by \(\mathcal{SP}_e(X, \bar{\tau}, \mathcal{A})\) (resp., \(\mathcal{SP}_e(C(X, \bar{\tau}, \mathcal{A})\) or \(\mathcal{SP}_e(O(X)\) (resp., \(\mathcal{SP}_e(C(X)\).

Lemma 2.2 [15] A soft set \((F, A)\) in a soft topological space \((X, \bar{\tau}, \mathcal{A})\) is soft \(p_e\)-open if and only if for each \(x_{e} \in (F, A)\), there exists a soft \(p_e\)-open set \((K, A)\) such that \(x_{e} \in (K, A) \subseteq (F, A)\).

Definition 2.16 [16] Let \((X, \bar{\tau}, \mathcal{A})\) be a soft topological space and let \((G, A)\) be a soft set. Then

1. A soft point \(x_{e} \in X\) is said to be a soft \(p_e\)-limit soft point of a soft set \((F, A)\), if for every soft \(p_e\)-open set \((G, A)\) containing \(x_{e}\), \((G, A) \cap [(F, A) \setminus \{x_{e}\}] \neq \emptyset\).

The set of all soft \(p_e\)-limit soft points of \((F, A)\) is called the soft \(p_e\)-derived set of \((F, A)\) and is denoted by \(\mathcal{SP}_e(D(F, A))\).

2. The soft \(p_e\)-closure of \((G, A)\) is the soft set

\[\mathcal{SP}_e(C)(G, A) = \overline{\{(K, B) \in \mathcal{SP}_e(C(X)); (G, A) \subseteq (K, B)\}\}.\]

3. The soft \(p_e\)-interior of \((G, A)\) is the soft set

\[\mathcal{SP}_e(int)(G, A) = \overline{\{(H, B) \in \mathcal{SP}_e(O(X); (H, B) \subseteq (G, A)\}\}}.

Lemma 2.3 [15] Let \((X, \bar{\tau}, \mathcal{A})\) be a soft topological space and let \((G, A)\) be a soft set. Then

\[\mathcal{SP}_e(C)(G, A) = (G, A) \cup \mathcal{SP}_e(D(F, A))\].

Lemma 2.4 [15] If \((F, A) \subseteq Y \subseteq X\) and \(\bar{Y}\) is soft clopen, then \((F, A) \subseteq \mathcal{SP}_e(O(Y)\) if and only if \((F, A) \subseteq \mathcal{SP}_e(O(Y)\).

Lemma 2.5 [15] Let \((F, A)\), \(\bar{Y} \subseteq X\) and \(\bar{Y}\) be soft clopen. If \((F, A) \subseteq \mathcal{SP}_e(O(X)\), then \((F, A) \cap \bar{Y} \subseteq \mathcal{SP}_e(O(Y)\).

Lemma 2.6 [16] Let \((F, A) \subseteq Y \subseteq X\). If \(\bar{Y}\) is soft clopen, then \(\mathcal{SP}_e(cl)(F, A) = \mathcal{SP}_e(cl)(F, A) \cap \bar{Y}\).

Definition 2.17 [17] Let \(\mathcal{SP}(X)_A\) and \(\mathcal{SP}(Y)_B\) be families of soft sets. Let \(u: X \rightarrow Y\) and \(p: E \rightarrow B\) be functions. Then, a function \(f_{pu}: \mathcal{SP}(X)_A \rightarrow \mathcal{SP}(Y)_B\) is defined as follows:

1. If \((F, A)\) is a soft set in \(\mathcal{SP}(X)_A\), then the image of \((F, A)\) under \(f_{pu}\), written as \(f_{pu}(F, A) = (f_{pu}(F), p(A))\), is a soft set in \(\mathcal{SP}(Y)_B\) such that

\[f_{pu}(F)(e') = \begin{cases} \bigcup_{e \in p^{-1}(e') \cap E} u(F(e)) & \text{if } p^{-1}(e') \cap E \neq \emptyset \\ \emptyset & \text{if } p^{-1}(e') \cap E = \emptyset \end{cases}\]

for all \(e' \in B\).
2. If \((G, B)\) is a soft set in \(SP(Y)_B\), then the inverse image of \((G, B)\) under \(f_{pu}\), written as \(f_{pu}^{-1}(G, B) = (f_{pu}^{-1}(G), p^{-1}(B))\), is a soft set in \(SP(X)_A\) such that

\[
f_{pu}^{-1}(G)(e) = \begin{cases} u^{-1}(G(p(e))) & \text{if } p(e) \in B \\ \phi & \text{otherwise} \end{cases}
\]

for all \(e \in A\).

The soft function \(f_{pu}\) is surjective, if \(p\) and \(u\) are surjective and it is injective if \(p\) and \(u\) are injective.

**Definition 2.18** Let \((X, \tilde{\tau}, A)\) and \((Y, \tilde{\mu}, B)\) be two soft topological spaces. A soft function \(f_{pu}: \tilde{X} \to \tilde{Y}\) is called soft continuous [18] (resp., \(\tilde{sp}_{c}\) -continuous [15]), if \(f_{pu}^{-1}((G, B)) \subseteq \tilde{\tau}\) (resp., \(f_{pu}^{-1}((G, B)) \subseteq \tilde{sp}_{c}(X)\)) for all \((G, B) \subseteq \tilde{\mu}\).

**Lemma 2.7** [15] A soft function \(f_{pu}: \tilde{X} \to \tilde{Y}\) is \(\tilde{sp}_{c}\)-continuous if and only if the inverse image of each soft open (soft closed) set is \(\tilde{sp}_{c}\)-open (\(\tilde{sp}_{c}\)-closed).

**Definition 2.19** [4] A soft subset \((F, A)\) of a soft topological space \((X, \tau, A)\) is said to be soft connected, if it does not have a soft separation, otherwise it is called soft disconnected.

**Definition 2.20** [4] A soft topological space \((X, \tau, A)\) is said to be a soft connected space, if it does not have a soft separation, otherwise it is called soft disconnected.

3. **\(\tilde{sp}_{c}\)-separated and \(\tilde{sp}_{c}\)-connected sets**

In this section, we introduce the concept of soft \(p_{c}\)-separated sets and soft \(p_{c}\)-connected sets in a soft topological space. Also, we discuss some of the main properties based on these concepts.

**Definition 3.1** Let \((X, \tau, A)\) be a soft topological space. Two non empty soft subsets \((F, A)\) and \((G, A)\) of \(SP(X)_A\) are said to be soft \(p_{c}\)-separated (\(\tilde{sp}_{c}\)-separated) sets over \(\tilde{X}\), if \(\tilde{sp}_{c}\text{cl}(F, A) \cap \tilde{sp}_{c}\text{cl}(G, A) = \phi\).

**Remark 3.1** Two soft sets \((F, A)\) and \((G, A)\) are \(\tilde{sp}_{c}\)-separated if and only if \((F, A)\) and \((G, A)\) are disjoint and neither of them contains \(\tilde{sp}_{c}\)-limit points of the other.

Therefore we have, if \((F, A) \cap \tilde{sp}_{c}\text{cl}(G, A) = \phi\) then \((F, A) \cap ((F, A) \cup \tilde{sp}_{c}\text{D}(G, A)) = [(F, A) \cap (G, A)] \cup [(F, A) \cap \tilde{sp}_{c}\text{D}(G, A)] = \phi\) so that \((F, A) \cap \tilde{sp}_{c}\text{D}(G, A) = \phi\). Therefore, \((F, A)\) contains no \(\tilde{sp}_{c}\)-limit points of \((G, A)\).

**Theorem 3.2** Let \((X, \tau, A)\) be a soft topological space. Then, the following are equivalent:

1. The only \(\tilde{sp}_{c}\text{-clopenn}\) set in \((X, \tau, A)\) is \(\tilde{X}\) and \(\phi\).
2. \(\tilde{X}\) is not the union of two disjoint non-empty \(\tilde{sp}_{c}\)-open sets.
3. \(\tilde{X}\) is not the union of two disjoint non-empty \(\tilde{sp}_{c}\)-closed sets.
4. \(\tilde{X}\) is not the union of two disjoint non-empty \(\tilde{sp}_{c}\)-separated sets.

**Proof.** (1) \(\Rightarrow\) (2). If \(\tilde{X} = (F, A) \cup (G, A)\), where \((F, A)\) and \((G, A)\) are disjoint non-empty \(\tilde{sp}_{c}\)-open sets, then, \(\tilde{X}\text{cl}(F, A) \cap (G, A)\) is a non-empty \(\tilde{sp}_{c}\)-closed set. Hence, \((G, A)\) is a non-empty proper \(\tilde{sp}_{c}\)-closed set. This contradicts (1), hence (2) is proved.

(2) \(\Rightarrow\) (3). Assume that \(\tilde{X}\) is not the soft union of two soft disjoint non-empty \(\tilde{sp}_{c}\)-open sets. Suppose that \(\tilde{X} = (K, A) \cup (L, A)\), where \((K, A)\) and \((L, A)\) are two disjoint non empty \(\tilde{sp}_{c}\)-closed sets. Now, \((K, A)\) and \((L, A)\) being respectively the complement of each other. Therefore, \((K, A)\) and \((L, A)\) are \(\tilde{sp}_{c}\)-open sets which contradict (2). Hence, we obtain (3).

(3) \(\Rightarrow\) (4). Suppose that \(\tilde{X} = (F, A) \cup (G, A)\), where \((F, A)\) and \((G, A)\) are non-empty \(\tilde{sp}_{c}\)-separated sets. Since \((F, A) \cap \tilde{sp}_{c}\text{cl}(G, A) = \phi\), we get \(\tilde{sp}_{c}\text{cl}(G, A) \subseteq \tilde{X}\text{cl}(F, A)\), hence \((G, A)\) is \(\tilde{sp}_{c}\)-closed set. Similarly, \((F, A)\) must be \(\tilde{sp}_{c}\)-closed set. This contradicts (3) and hence the proof is finished.

(4) \(\Rightarrow\) (1). Suppose that \(\tilde{X} = (F, A)\) is a non-empty proper \(\tilde{sp}_{c}\)-closed subset of \(\tilde{X}\). Then, \((G, A) = \tilde{X}\text{cl}(F, A)\) is a non-empty proper \(\tilde{sp}_{c}\)-closed subset of \(\tilde{X}\). Since \(\tilde{X} = (F, A) \cup (G, A)\), so \((F, A)\) and \((G, A)\) are \(\tilde{sp}_{c}\)-separated sets, which shows that (4) is false. Therefore, (1) is proved.

**Definition 3.3** A soft \(p_{c}\)-separation (\(\tilde{sp}_{c}\)-separation) of a soft topological space \((X, \tau, A)\) is a pair of \(\tilde{sp}_{c}\)-separated sets \((F, A)\) and \((G, A)\) whose union is \(\tilde{X}\).

The following example illustrates a non-trivial \(\tilde{sp}_{c}\)-separation of a soft topological space \((X, \tau, A)\).
Example 3.1 Let $X = \{x, y\}$, $A = \{e_1, e_2\}$. Let $(X, \tau, A$) be the soft topological space where $\tilde{\tau} = \{\tilde{X}, \tilde{\phi}, (F_1, A), (F_2, A), (F_3, A), (F_4, A)\}$ where

$$(F_1, A) = \{(e_1, \{x, y\}, \{e_2, \phi\}\}, (F_2, A) = \{(e_1, \{x\}, \{e_2, \{x\}\}\}, (F_3, A) = \{(e_1, \phi, \{x, y\}\}, (F_4, A) = \{(e_1, \phi, \{e_2, \phi\}\}\}.\]

Then, it is easy to check that $\tilde{sp}_O(X) = \{\tilde{X}, \tilde{\phi}, (F_1, A), (F_3, A)\}$ and $\tilde{sp}_O((F_3, A) \cap (F_2, A) = \tilde{sp}_O((F_3, A) \cap (F_2, A) = \phi$. Therefore, $(F_1, A)$ and $(F_3, A)$ are $\tilde{sp}_O$-closed sets. Hence $(F_1, A)$ and $(F_3, A)$ form an $\tilde{sp}_O$-separation of $\tilde{X}$.

Theorem 3.4 Let $(F, A)$ and $(G, A)$ be two non empty soft sets in a space $\tilde{X}$. Then, the following statements are true:

1. If $(F, A)$ and $(G, A)$ are $\tilde{sp}_O$-closed and $(F_1, A) \subseteq (F, A)$, $(G_1, A) \subseteq (G, A)$, then $(F_1, A)$ and $(G_1, A)$ are also $\tilde{sp}_O$-closed.
2. (3) If $F_i(A) \cap \tilde{G}(A) = \phi$ such that each of $(F, A)$ and $(G, A)$ are both $\tilde{sp}_O$-closed (if they are $\tilde{sp}_O$-open and their union is $\tilde{X}$), then $(F, A)$ and $(G, A)$ are $\tilde{sp}_O$-closed.
3. If each of $(F, A)$ and $(G, A)$ are both $\tilde{sp}_O$-closed ($\tilde{sp}_O$-open) and if $(U, A) = (F, A) \tilde{\cap} (X \setminus (G, A))$, $(V, A) = (G, A) \tilde{\cap} (X \setminus (F, A))$, then $(U, A)$ and $(V, A)$ are $\tilde{sp}_O$-closed.

Proof. (1) Since $(F_1, A) \subseteq (F, A)$, then $\tilde{sp}_O((F_1, A) \subseteq (F_3, A) \tilde{\cap} (F, A)$. So, $\tilde{sp}_O((F, A) \tilde{\cap} (F, A) = \phi$ and $\tilde{sp}_O((F, A) \tilde{\cap} (G, A) = \phi$. Similarly $(F_1, A) \tilde{\cap} (F, A) = \phi$ and $(G_1, A)$ are $\tilde{sp}_O$-closed.

(2) Since $(F, A) = (F_1, A) \tilde{\cap} (F, A)$, $(G, A) = (F_2, A) \tilde{\cap} (G, A)$, and $(F, A) \tilde{\cap} (G, A) = \phi$, then $\tilde{sp}_O((F, A) \tilde{\cap} (F, A) = \phi$ and $\tilde{sp}_O((G, A) \tilde{\cap} (F, A) = \phi$. Hence $(F, A)$ and $(G, A)$ are $\tilde{sp}_O$-closed. If $(F, A)$ and $(G, A)$ are $\tilde{sp}_O$-open and their union is $\tilde{X}$, then their complements are $\tilde{sp}_O$-closed with empty intersection.

Theorem 3.5 The soft sets $(F, A)$ and $(G, A)$ of a space $\tilde{X}$ are $\tilde{sp}_O$-closed if and only if there exist $\tilde{sp}_O$-open sets $(U, A)$ and $(V, A)$ such that $(F, A) \subseteq (U, A)$, $(G, A) \subseteq (V, A)$ and $(F, A) \tilde{\cap} (U, A) = \phi$, $(G, A) \tilde{\cap} (U, A) = \phi$.

Proof. Let $(F, A)$ and $(G, A)$ be $\tilde{sp}_O$-closed sets. Then, the sets $(V, A) = \tilde{X} \setminus \tilde{sp}_O((F, A) \tilde{\cap} (U, A), (G, A) \tilde{\cap} (V, A)$ and $(F, A) \tilde{\cap} (U, A) = \phi$. Conversely, let $(U, A), (V, A) \subseteq (F, A) \tilde{\cap} (U, A), (G, A) \tilde{\cap} (V, A)$ and $(F, A) \tilde{\cap} (U, A) = \phi$. Since $(V, A)$ and $(X \setminus (U, A))$ are $\tilde{sp}_O$-closed, then $\tilde{sp}_O((F, A) \tilde{\cap} (U, A) \subseteq (X \setminus (F, A) \tilde{\cap} (U, A)) \subseteq (X \setminus (F, A) \tilde{\cap} (U, A)) = \phi$. Thus, $\tilde{sp}_O((F, A) \tilde{\cap} (U, A) = \phi$ and hence the proof is finished.

Theorem 3.6 Let $(F, A)$ and $(G, A)$ be nonempty soft disjoint subsets of a space $\tilde{X}$ and $(V, A) = (G, A) \cup (F, A)$. Then $(F, A)$ and $(G, A)$ are $\tilde{sp}_O$-closed if and only if each of $(F, A)$ and $(G, A)$ is $\tilde{sp}_O$-closed in $(V, A)$.

Proof. Let $(F, A)$ and $(G, A)$ be $\tilde{sp}_O$-closed sets. By Remark 3.1, $(F, A)$ contains no $\tilde{sp}_O$-limit points of $(G, A)$. Then, $(G, A)$ contains all $\tilde{sp}_O$-limit points of $(G, A)$ which implies that $(G, A)$ is $\tilde{sp}_O$-closed in $(G, A) \cup (F, A) = (V, A)$. Similarly $(F, A)$ is $\tilde{sp}_O$-closed in $(V, A)$. The converse part is obvious.

Theorem 3.7 If $(F, A)$ is an $\tilde{sp}_O$-connected set and $(F, A) \subseteq (G, A) \subseteq \tilde{sp}_O((F, A)$, then $(G, A)$ is $\tilde{sp}_O$-connected.

Proof. Let $(F, A) \subseteq \tilde{sp}_O((F, A)$ be an $\tilde{sp}_O$-connected set and $(G, A)$ be any soft subset of $\tilde{X}$ such that $(F, A) \subseteq (G, A) \subseteq \tilde{sp}_O((F, A)$ We have to show that $(G, A)$ is an $\tilde{sp}_O$-connected set. Suppose that $(G, A)$ is not $\tilde{sp}_O$-connected. Then, there exists a pair of $\tilde{sp}_O$-connected sets $(F_1, A)$ and $(F_2, A)$ such that $(G, A) = (F_1, A) \tilde{\cup} (F_2, A)$. Suppose that $(F, A) \tilde{\cap} (F_1, A) \neq \emptyset$ and $(F, A) \tilde{\cap} (F_2, A) \neq \emptyset$. Then, $(F, A) = ((F, A) \tilde{\cap} (F_1, A)) \tilde{\cup} ((F, A) \tilde{\cap} (F_2, A))$. But $(F, A) \tilde{\cap} (F_1, A)$ and $(F, A) \tilde{\cap} (F_2, A)$ are $\tilde{sp}_O$-closed sets. This is a contradiction to the $\tilde{sp}_O$-connectedness of $(F, A)$. Hence, either $(F, A) \subseteq (F_1, A)$ or $(F, A) \subseteq (F_2, A)$.
(F, A) $\sqsubseteq$ (F_1, A), then $\tilde{sp}_c cl (F, A) \sqsubseteq \tilde{sp}_c cl (F_1, A)$. Since (F_1, A) and (F_2, A) are $\tilde{sp}_c$-separated sets, then $\tilde{sp}_c cl (F_1, A) \cap (F_2, A) = \emptyset$. Therefore, $\tilde{sp}_c cl (F, A) \cap (F_2, A) = \emptyset$ but $(F_2, A) \sqsubseteq (G, A)$. Then, by the hypothesis, we have $(F_2, A) \sqsubseteq (G, A) \sqsubseteq \tilde{sp}_c cl (F, A)$. Therefore, $\tilde{sp}_c cl (F, A) \cap (F_2, A) = (F_2, A)$. Thus, $\tilde{sp}_c cl (F, A) \cap (F_2, A) = \emptyset$ and $\tilde{sp}_c cl (F, A) \cap (F_2, A) = (F_2, A)$. Hence, $(F_2, A) = \emptyset$, which is a contradiction. Similarly, if $(F, A) \sqsubseteq (F_2, A)$, then we obtain that $(F_1, A) = \emptyset$, which is a contradiction. Therefore, there does not exist an $\tilde{sp}_c$-separation of $(G, A)$. Hence, $(G, A)$ is $\tilde{sp}_c$-connected.

**Theorem 3.8** Let $(Y, \tau_Y, A)$ be a soft subspace of a soft topological space $(X, \tau, A)$ and $(F, A) \sqsubseteq Y \sqsubseteq X$. Then, $(F, A)$, $(G, A)$ are soft $\tilde{sp}_c$-separated in $Y$ if and only if $(F, A)$, $(G, A)$ are soft $\tilde{sp}_c$-separated in $X$.

**Proof.** $(F, A)$ and $(G, A)$ are $\tilde{sp}_c$-separated in $Y$ $\iff$ $\tilde{sp}_c cl_Y (F, A) \cap (G, A) = \emptyset$ $\iff$ $[\tilde{sp}_c cl_X (F, A) \cap (G, A) = \emptyset]$ $\iff$ $\tilde{sp}_c cl_X (F, A) \cap (G, A) = \emptyset$ and $\tilde{sp}_c cl_X (G, A) \cap (F, A) = \emptyset$. $\iff$ $(F, A)$ and $(G, A)$ are $\tilde{sp}_c$-separated in $X$.

**4 $\tilde{sp}_c$-connectedness**

In this section, we introduce the concept of $\tilde{sp}_c$-connected spaces by using $\tilde{sp}_c$-open and $\tilde{sp}_c$-closed sets in soft topological spaces. Basic properties of this space are obtained.

**Definition 4.1** Let $(X, \tau, A)$ be a soft topological space. A soft set $(F, A) \in SP (X)_A$ is said to be $\tilde{sp}_c$-connected if it does not have an $\tilde{sp}_c$-separation, otherwise it is called $\tilde{sp}_c$-disconnected.

**Remark 4.2** The following statements are obvious.

1. The singleton soft set in a soft topological space is an $\tilde{sp}_c$-connected set.
2. All soft subsets in the soft indiscrete topological space are $\tilde{sp}_c$-connected.

**Definition 4.3** A soft subset $(F, A)$ of a soft topological space $(X, \tau, A)$ is said to be soft pre-connected if it does not have a soft pre-separation, otherwise it is called soft pre-disconnected.

**Proposition 4.4** Every soft pre-connected set is an $\tilde{sp}_c$-connected set.

**Proof.** Let $(F, A)$ be a soft pre-connected set in the soft topological space $(X, \tau, A)$. Hence there does not exist a soft pre-separation of $(F, A)$. Since every $\tilde{sp}_c$-open set is a soft pre-open set, hence there does not exist an $\tilde{sp}_c$-separated set of $(F, A)$. So, $(F, A)$ is $\tilde{sp}_c$-connected.

If the soft topological space is a soft $T_1$, then by Proposition 2.14, the converse of Proposition 3.9 is also true.

An $\tilde{sp}_c$-connected set needs not be a soft pre-connected set as it is shown in the following example:

**Example 4.1** Any subset in the indiscrete soft topological space is $\tilde{sp}_c$-connected and since the family of soft pre-open sets forms the discrete soft topology, then any soft subset containing more than one soft point is not soft pre-connected.

**Theorem 4.5** Let $(X, \tau, A)$ be a soft topological space and $(F, A)$ be an $\tilde{sp}_c$-connected set. Let $(F_1, A)$ and $(F_2, A)$ be $\tilde{sp}_c$-separated sets. If $(F, A) \sqsubseteq (F_1, A) \cup (F_2, A)$, then either $(F, A) \sqsubseteq (F_1, A)$ or $(F, A) \sqsubseteq (F_2, A)$.

**Proof.** If $(F, A) \sqsubseteq (F_1, A)$ and $(F, A) \sqsubseteq (F_2, A)$, then $(G, A) = (F, A) \cap (F_1, A) \neq \emptyset$, $(H, A) = (F, A) \cap (F_2, A) \neq \emptyset$, and $(F, A) = (G, A) \cup (H, A)$. Since $(G, A) \sqsubseteq (F_1, A)$, implies that $\tilde{sp}_c cl (G, A) \sqsubseteq \tilde{sp}_c cl (F_1, A)$. Since $(F_1, A)$, $(F_2, A)$ are $\tilde{sp}_c$-separated, we have $\tilde{sp}_c cl (F_1, A) \cap (F_2, A) = \emptyset$. Therefore, $\emptyset = \tilde{sp}_c cl (F_1, A) \cap (F_2, A) \sqsubseteq \tilde{sp}_c cl (G, A) \cap (F_2, A) \sqsubseteq \tilde{sp}_c cl (G, A) \cap (H, A)$. Hence, $\tilde{sp}_c cl (G, A) \cap (H, A) = \emptyset$. By the same way, we can show that $(G, A) \cap \tilde{sp}_c cl (H, A) = \emptyset$, but $(F, A) = (G, A) \cup (H, A)$, hence there exists an $\tilde{sp}_c$-separation of $(F, A)$. Therefore, $(F, A)$ is not an $\tilde{sp}_c$-connected set, which is a contradiction. Hence, either $(F, A) \sqsubseteq (F_1, A)$ or $(F, A) \sqsubseteq (F_2, A)$.

**Corollary 4.6** If $(F, A)$ is an $\tilde{sp}_c$-connected set, then $\tilde{sp}_c cl (F, A)$ is $\tilde{sp}_c$-connected.

**Proof.** The proof follows directly from Theorem 3.7.

**Theorem 4.7** The soft union $(F, A)$ of any family $\{(F_i, A) : i \in I\}$ of $\tilde{sp}_c$-connected sets having a nonempty soft intersection is an $\tilde{sp}_c$-connected set.

**Proof.** Let $(F, A)$ be a soft union of any family of $\tilde{sp}_c$-connected sets having a non-empty soft intersection. Suppose that $(F, A) = (H_1, A) \cup (H_2, A)$, where $(H_1, A)$ and $(H_2, A)$ form an $\tilde{sp}_c$-separation of $(F, A)$. By hypothesis, we may choose a soft point $x_\alpha \in \bigcap_{i \in I} (F_i, A)$. 

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Then, \( x_i \in (F_i, A) \) for all \( i \in I \). So \( x_i \in (F, A) \), and then either \( x_i \in (H_1, A) \) or \( x_i \in (H_2, A) \). Since \((H_1, A) \) and \((H_2, A) \) are soft disjoint, we must have \((F_i, A) \subseteq (H_1, A) \) (say), since \((F_i, A) \) is \( \mathcal{S}p_c \)-connected and it is true for all \( i \in I \), and so \( (F, A) \subseteq (H_1, A) \). From this we obtain that \((H_2, A) = \emptyset \), which is a contradiction.

**Definition 4.8** A soft topological space \((X, \tau, A)\) is said to be a \( \mathcal{S}p_c \)-connected space if it does not have an \( \mathcal{S}p_c \)-separation, otherwise it is called \( \mathcal{S}p_c \)-disconnected.

It is clear that every indiscrete topological space with at least two soft points is an \( \mathcal{S}p_c \)-connected space and is soft pre-disconnected.

**Definition 4.9** A soft topological space \((X, \tau, A)\) is said to be a soft pre-connected space if it does not have a soft pre-separation, otherwise it is called soft pre-disconnected.

**Proposition 4.10** Every soft pre-connected space is an \( \mathcal{S}p_c \)-connected. Furthermore, the converse is also true if the space is soft \( T_1 \).

**Corollary 4.11** A soft topological space \((X, \tau, A)\) is \( \mathcal{S}p_c \)-disconnected if and only if there exists a non-empty proper soft subset of \( X \) which is \( \mathcal{S}p_c \)-clopen.

**Proof.** It follows directly from (1) and (4) of Theorem 3.2.

**Proposition 4.12** If \((X, \tau, A)\) is a soft disconnected space, then it is \( \mathcal{S}p_c \)-disconnected.

**Proof.** If \((X, \tau, A)\) is a soft disconnected space, then it contains a non-empty proper soft clopen set \((F, A)\), so \((F, A)\) is both \( \mathcal{S}p_c \)-open and \( \mathcal{S}p_c \)-closed. Hence, by Corollary 4.11, \((X, \tau, A)\) is \( \mathcal{S}p_c \)-disconnected.

**Proposition 4.13** Let \((X, \tau, A)\) be a finite soft topological space. If \((X, \tau, A)\) contains a non-empty proper \( \mathcal{S}p_c \)-open set, then \((X, \tau, A)\) is a soft disconnected space and hence, \( \mathcal{S}p_c \)-disconnected.

**Proof.** In a finite space \((X, \tau, A)\), if \((F, A)\) is a non-empty proper \( \mathcal{S}p_c \)-open set, then \((F, A)\) is a union of soft closed sets and hence it is closed. Also \((F, A)\) is pre-open, implies that \((F, A) \subseteq \mathcal{S}int \mathcal{S}cl(F, A)\), so \((F, A) \subseteq \mathcal{S}int(F, A)\) which implies that \((F, A)\) is soft open. Therefore, \((X, \tau, A)\) is soft disconnected and by Proposition 4.12 it is \( \mathcal{S}p_c \)-disconnected.

**Corollary 4.14** A finite soft space \((X, \tau, A)\) is \( \mathcal{S}p_c \)-disconnected (\( \mathcal{S}p_c \)-connected) if and only if it is soft disconnected (soft connected).

**Proof.** Follows from Proposition 4.12 and Proposition 4.13.

The following example shows that a soft connected space may not be \( \mathcal{S}p_c \)-connected.

**Example 4.2** Let \( \mathbb{R} \) be the set of real numbers and \( A = \{e_1, e_2\} \) and let \( \tilde{\tau} \) be a family consisting of \( \tilde{\phi} \) and all soft subsets \((F, A)\) such that \( F(e_1) = B \) where \( \mathbb{R}\setminus B \) is finite and \( F(e_2) = D \) where \( D \subseteq \mathbb{R} \). Then, this space is soft connected but it is \( \mathcal{S}p_c \)-disconnected because the soft sets \((G, A)\) such that \( G(e_1) = \mathbb{Q} \), \( F(e_2) = D \), where \( D \subseteq \mathbb{R} \) and \((H, A)\) such that \( H(e_1) = \mathbb{R}\setminus \mathbb{Q} \) and \( F(e_2) = D^c \), where \( D \subseteq \mathbb{R} \) are disjoint \( \mathcal{S}p_c \)-open and \( \mathcal{S}p_c \)-closed and their union is \( \mathbb{R} \).

In Proposition 4.13, the condition of the soft topological space to be finite is necessary. The following example shows that there exists a non-empty proper \( \mathcal{S}p_c \)-open set in the space but the space is neither soft disconnected nor \( \mathcal{S}p_c \)-disconnected.

**Example 4.3** Let \( X \) be any infinite set, \( A \) consists of infinite parameters, \( P_e \) is a fixed soft point in \( X \), and let \( \tilde{\tau} \) be the family of all soft subsets \((F, A)\) such that either \( P_e \notin (F, A) \) or, if \( P_e \in (F, A) \), then \( \bigcup_{e \in E} X \setminus F(e) \) is finite. Then, this space contains a non-empty proper \( \mathcal{S}p_c \)-open set but it is neither soft disconnected nor \( \mathcal{S}p_c \)-disconnected.

In the following example we show that it is not necessarily true that if the space is \( \mathcal{S}p_c \)-connected then every subspace must be \( \mathcal{S}p_c \)-connected.

**Example 4.4** Let \( X = \{x, y\} \) and \( A = \{e_1, e_2\} \). Let \( \tilde{\tau} = \{\tilde{X}, \tilde{\phi}, (F_1, A), (F_2, A), (F_3, A), (F_4, A)\} \) be the soft topology where
\[
(F_1, A) = \{(e_1, \{x\}), (e_2, \{\phi\})\},
(F_2, A) = \{(e_1, \{x\}), (e_2, \{y\})\},
(F_3, A) = \{(e_1, X), (e_2, \{y\})\},
(F_4, A) = \{(e_1, \{\phi\}), (e_2, \{y\})\}.
\]
This space is both soft connected and \( \mathcal{S}p_c \)-connected. Consider the soft subset \((F_2, A) = \{(e_1, \{x\}), (e_2, \{y\})\}\). Then the soft relative topology induced on the soft set \((F_2, A)\) is
\[
\tau_{(F_2, A)} = \{(\emptyset, (F_2, A), (F_1, A), (F_4, A))\}.
\]
Then, \( \mathcal{S}p_c O((F_5, A)) = \tau_{(F_2, A)} \). Therefore, the soft set \((F_2, A)\) has an \( \mathcal{S}p_c \)-separation in the soft relative topology induced on the soft subset \((F_2, A)\). Hence, \((F_2, A)\) is an \( \mathcal{S}p_c \)-disconnected subset of a soft topological space \((X, \tau, A)\).
Corollary 4.15 A soft topological space \((X, \tau, A)\) is an \(\tilde{sp}_c\)-disconnected space if and only if any one of the following statements is satisfied.

1. \(\tilde{X}\) is a soft union of two non empty disjoint \(\tilde{sp}_c\)-open sets.
2. \(\tilde{X}\) is a soft union of non empty disjoint \(\tilde{sp}_c\)-closed sets.

Proof. It follows directly from parts 1, 2, and 3 of Theorem 3.2.

Theorem 4.16 Let \((X, \tau_1, A)\) be a \(\tilde{sp}_c\)-connected space and \(\tau_2 \subseteq \tau_1\). Then, \((X, \tau_2, A)\) is \(\tilde{sp}_c\)-connected.

Proof. Suppose on the contrary that \((X, \tau_2, A)\) is not \(\tilde{sp}_c\)-connected, and let \((H, A)\) and \((G, A)\) be an \(\tilde{sp}_c\)-separation of \((X, \tau_2, A)\). Since \(\tilde{\tau}_2 \subseteq \tilde{\tau}_1\) then \((H, A)\) and \((G, A)\) are \(\tilde{sp}_c\)-separation of \((X, \tau_1, A)\), which is a contradiction. Therefore, \((X, \tau_2, A)\) is \(\tilde{sp}_c\)-connected.

Theorem 4.17 Let \((X, \tau_1, A)\) and \((Y, \tau_2, B)\) be two soft topological spaces and \(u : X \rightarrow Y\) and \(p : A \rightarrow B\) be functions. Also a soft function \(f_{pu} : SP(X)_A \rightarrow SP(Y)_B\) is \(\tilde{sp}_c\)-continuous and onto. If \((X, \tau_1, A)\) is \(\tilde{sp}_c\)-connected, then the soft image \((Y, \tau_2, B)\) is soft connected.

Proof. Let a soft function \(f_{pu} : SP(X)_A \rightarrow SP(Y)_B\) be \(\tilde{sp}_c\)-continuous and onto. On the contrary, suppose that \((Y, \tau_2, B)\) is soft disconnected and let \((H, B)\) be a soft non-empty proper subset of \((Y, \tau_2, B)\) which is both soft open and soft closed. Since \(f_{pu} : SP(X)_A \rightarrow SP(Y)_B\) is soft \(\tilde{sp}_c\)-continuous. Therefore, by Lemma 2.7, \(f_{pu}^{-1}(H, B)\) is both \(\tilde{sp}_c\)-open and \(\tilde{sp}_c\)-closed in \((X, \tau_1, A)\), which is a contradiction. Hence \((Y, \tau_2, B)\) is soft connected.

Definition 4.18 Let \((X, \tau, A)\) be a soft topological space and \(x_\alpha \in (F, A) \in SP(X)_A\). The \(\tilde{sp}_c\)-component of \((F, A)\) corresponding to \(x_\alpha\) is the union of all \(\tilde{sp}_c\)-connected subsets in \((F, A)\) containing \(x_\alpha\).

From Theorem 4.7, we obtain that the \(\tilde{sp}_c\)-component of \(X\) is \(\tilde{sp}_c\)-connected.

Theorem 4.19 For a topological space \((X, \tau, A)\), the following properties hold:

1. Each \(\tilde{sp}_c\)-component of \(X\) is a maximal \(\tilde{sp}_c\)-connected subset of \(X\).
2. The set of all distinct \(\tilde{sp}_c\)-components of \(X\) forms a partition of \(X\).
3. Each \(\tilde{sp}_c\)-component of \(X\) is \(\tilde{sp}_c\)-closed in \(X\).

Proof. (1) Obvious.

2. Each soft point \(x_\alpha \in SP(X)_A\) is contained in an \(\tilde{sp}_c\)-component of \(X\) containing \(x_\alpha\). Suppose that \((F_1, A)\) and \((F_2, A)\) are two distinct \(\tilde{sp}_c\)-components of \(X\). If \((F_1, A)\) and \((F_2, A)\) intersect, then by Theorem 3.13, \((F_1, A) \cap (F_2, A)\) is \(\tilde{sp}_c\)-connected. Thus, either \((F_1, A)\) is not maximal or \((F_2, A)\) is not maximal, which is a contradiction. Therefore, \((F_1, A)\) and \((F_2, A)\) are disjoint.

3. Let \((F, A)\) be any \(\tilde{sp}_c\)-component of \(X\) containing \(x_\alpha\). By Corollary 4.6, \(\tilde{sp}_c cl(F, A)\) is \(\tilde{sp}_c\)-connected set containing \(x_\alpha\). Since \((F, A)\) is maximal \(\tilde{sp}_c\)-connected set containing \(x_\alpha\), so \(\tilde{sp}_c cl(F, A) \subseteq (F, A)\). Thus, \((F, A)\) is \(\tilde{sp}_c\)-closed in \(\tilde{X}\).

Theorem 4.20 A soft topological space \((X, \tau, A)\) is \(\tilde{sp}_c\)-disconnected if and only if there is a non-empty proper subset which has an empty soft \(\tilde{sp}_c\)-boundary.

Proof. Let \((X, \tau, A)\) be soft \(\tilde{sp}_c\)-disconnected. Then by Corollary 4.11, \((X, \tau, A)\) has a proper \(\tilde{sp}_c\)-clopen soft set \((F, A)\). Thus, \(\tilde{sp}_c cl(F, A) = (F, A) = \tilde{sp}_c int(F, A) = \tilde{X} \setminus (\tilde{sp}_c clopen(\tilde{X} \setminus (F, A)))\). Hence, \(\tilde{sp}_c bd(F, A) = \tilde{sp}_c clopen((F, A)) \cap \tilde{sp}_c clopen(\tilde{X} \setminus (F, A)) = \emptyset\). Therefore, \((F, A)\) has an empty soft \(\tilde{sp}_c\)-boundary.

Conversely, suppose that there is a non-empty proper soft subset \((F, A)\) that has an empty soft \(\tilde{sp}_c\)-boundary. Then, \(\tilde{sp}_c bd(F, A) = \tilde{sp}_c clopen(F, A) \cap \tilde{sp}_c clopen(\tilde{X} \setminus (F, A)) = \emptyset\). Consequently, \(\tilde{sp}_c cl(F, A) = \tilde{X} \setminus \tilde{sp}_c clopen(\tilde{X} \setminus (F, A)) \subseteq \tilde{sp}_c int(F, A) \subseteq (F, A)\). Hence, \((F, A)\) is a proper \(\tilde{sp}_c\)-clopen soft set and by Corollary 4.11, \((X, \tau, A)\) is \(\tilde{sp}_c\)-disconnected.

Definition 4.21 A space \((X, \tau, A)\) is called locally \(\tilde{sp}_c\)-connected at \(x_\alpha \in SP(X)_A\), if for each \(\tilde{sp}_c\)-open set \((G, A)\) containing \(x_\alpha\), there is an \(\tilde{sp}_c\)-connected \(\tilde{sp}_c\)-open set \((H, A)\) such that \(x_\alpha \in (H, A) \subseteq (G, A)\). The space \(X\) is locally \(\tilde{sp}_c\)-connected if it is locally \(\tilde{sp}_c\)-connected at each of its soft points.

It is clear that the discrete soft space is locally \(\tilde{sp}_c\)-connected but it is not \(\tilde{sp}_c\)-connected.
Theorem 4.22 A space $(X, \tau, A)$ is locally $\tilde{\mathcal{S}} p_c$-connected if and only if the $\tilde{\mathcal{S}} p_c$-components of each $\tilde{\mathcal{S}} p_c$-open subset of $X$ are $\tilde{\mathcal{S}} p_c$-open.

Proof. Suppose that $X$ is locally $\tilde{\mathcal{S}} p_c$-connected. $(G, A)$ is an $\tilde{\mathcal{S}} p_c$-open subset of $X$ and $(F, A)$ is an $\tilde{\mathcal{S}} p_c$-component of the soft subset $(G, A)$ corresponding to the soft point $x_\alpha$. Then, by definition, there is an $\tilde{\mathcal{S}} p_c$-open set $(H, A) \subseteq X$ such that $x_\alpha \in (H, A) \subseteq (G, A)$. Since $(F, A)$ is an $\tilde{\mathcal{S}} p_c$-component of $(G, A)$, so we get $x_\alpha \in (H, A) \subseteq (F, A)$. Thus, by Lemma 2.2, $(F, A)$ is an $\tilde{\mathcal{S}} p_c$-open set.

Conversely, let $(G, A) \subseteq X$ be an $\tilde{\mathcal{S}} p_c$-open set and $x_\alpha \in (G, A)$. By the hypothesis, the $\tilde{\mathcal{S}} p_c$-component $(H, A)$ of $(G, A)$ containing $x_\alpha$ is $\tilde{\mathcal{S}} p_c$-open, so $X$ is locally $\tilde{\mathcal{S}} p_c$-connected at $x_\alpha$.

Theorem 4.23 A soft topological space $(X, \tau, A)$ is locally $\tilde{\mathcal{S}} p_c$ connected if and only if, given any soft point $x_\alpha \in SP(X)_A$ and a $\tilde{\mathcal{S}} p_c$-open set $(G, A)$ containing $x_\alpha$, there is a soft open set $(H, A)$ containing $x_\alpha$ such that $(H, A)$ is contained in a single $\tilde{\mathcal{S}} p_c$-component of $(G, A)$.

Proof. Let $X$ be locally $\tilde{\mathcal{S}} p_c$-connected, $x_\alpha \in SP(X)_A$ and $(G, A)$ be an $\tilde{\mathcal{S}} p_c$-open set containing $x_\alpha$. Let $(F, A)$ be the $\tilde{\mathcal{S}} p_c$ component of $(G, A)$ that contains $x_\alpha$. Since $X$ is locally $\tilde{\mathcal{S}} p_c$ connected and $(G, A)$ is $\tilde{\mathcal{S}} p_c$ open, there is an $\tilde{\mathcal{S}} p_c$ connected $\tilde{\mathcal{S}} p_c$-open set $(H, A)$ such that $x_\alpha \in (H, A) \subseteq (G, A)$. By Theorem 4.14, $(F, A)$ is the maximal $\tilde{\mathcal{S}} p_c$ connected set containing $x_\alpha$ and so $x_\alpha \in (H, A) \subseteq (F, A) \subseteq (G, A)$. Since $\tilde{\mathcal{S}} p_c$ components are disjoint sets, it follows that $(H, A)$ is not contained in any other $\tilde{\mathcal{S}} p_c$ component of $(G, A)$.

Conversely, we suppose that, given any soft point $x_\alpha \in SP(X)_A$ and any $\tilde{\mathcal{S}} p_c$-open set $(G, A)$ containing $x_\alpha$, there is a soft open set $(H, A)$ containing $x_\alpha$ which is contained in a single $\tilde{\mathcal{S}} p_c$-component $(F, A)$ of $(G, A)$. Then $x_\alpha \in (H, A) \subseteq (F, A) \subseteq (G, A)$. Let $x_{\alpha 1} \in (F, A)$, then $x_{\alpha 1} \in (G, A)$. Thus there is a soft open set $(F_2, A)$ such that $x_{\alpha 1} \in (F_2, A)$ and $(F_2, A)$ is contained in a single $\tilde{\mathcal{S}} p_c$-component of $(G, A)$. As the $\tilde{\mathcal{S}} p_c$ components are disjoint soft sets and $x_{\alpha 1} \in (F_1, A)$, hence $x_{\alpha 1} \in (F_2, A) \subseteq (F_1, A)$. Thus, $(F_1, A)$ is soft open. Hence, for every $x_\alpha \in SP(X)_A$ and for every $\tilde{\mathcal{S}} p_c$-open set $(G, A)$ containing $x_\alpha$, there is an $\tilde{\mathcal{S}} p_c$-connected $\tilde{\mathcal{S}} p_c$-open set $(F_1, A)$ such that $x_\alpha \in (F_1, A) \subseteq (G, A)$. Thus, $X$ is locally $\tilde{\mathcal{S}} p_c$-connected at $x_\alpha$. Since $x_\alpha \in SP(X)_A$ is arbitrary, so $X$ is locally $\tilde{\mathcal{S}} p_c$-connected.

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