Approximate Solutions for Systems of Volterra Integro-differential Equations Using Laplace–Adomian Method

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Abstract
Some modified techniques are used in this article in order to have approximate solutions for systems of Volterra integro-differential equations. The suggested techniques are the so called Laplace-Adomian decomposition method and Laplace iterative method. The proposed methods are robust and accurate as can be seen from the given illustrative examples and from the comparison that are made with the exact solution.

Keywords: Volterra integro-differential equations; Laplace transform; Adomian decomposition method.

Introduction
Integro-differential equations emerge in different branches of physics and engineering, such as fluid mechanics, thin films, diffusion processes and so on [1-3]. It has encouraged many authors to have numerical and approximate solutions [4-6]. The Laplace-Adomian decomposition method (LADM) combines between two powerful methods, which are the Laplace transform and the Adomian decomposition, and was introduced for the first time by Khuri [7, 8]. Previously presented methods handled a wide class of non-linear problems and showed a great success in terms of obtaining the approximate results. Agadjanov [9] solved the Duffing equation by using LADM, while Hossein zadeh et al. applied LADM for solving Klein–Gordon equation [10]. Khan et al. employed LADM to obtain a solution for nonlinear coupled partial differential equations [11]. Jafarit al. solved non-linear fractional diffusion–wave equations by using LADM [12]. Manafianheris [13] applied the modified LADM for solving integro-differential equations. In this work we employ two techniques to obtain an
approximate solution for systems of integro-differential equations, which are LADM and Laplace Iterative Method (LIM). This article is ordered as follows: The description of LADM is illustrated in section 1, in section 2 Laplace Iterative Method is given, in section 3 some illustrative examples are given, and at last the conclusions are presented in section 4.

1. Laplace-Adomian Decomposition Method (LADM)
Consider the system of Volterra’s integro-differential equations:

\[ y_i^{(n)}(t) = f_i(t) + \int_{0}^{t} k_i(x,t)F(y_1,y_2,\ldots,y_m) \, dt, \quad i = 1,2,\ldots,m \]  

where \( c_{ij}, j=0,1,2,\ldots,m-1 \) are given constants.

The technique consists of implementing Laplace transformation to equation (1), hence

\[ L(y_i^{(n)}) = F_i(s) + L(\int_{0}^{t} k_i(x,t)F(y_1,y_2,\ldots,y_m) \, dt), \quad i = 1,2,\ldots,m \]  

Equation (2) can be simplified as:

\[ L(y_i^{(n)}) = F_i(s) + L(\int_{0}^{t} k_i(x,t)F(y_1,y_2,\ldots,y_m) \, dt), \quad i = 1,2,\ldots,m \]  

According to the properties of the Laplace transform we get:

\[ s^nY_i(s) - s^{n-1}y_i(0) - s^{n-2}y_i'(0) - \ldots - y_i^{(n-1)}(0) = \ 
\]

\[ F_i(s) + L(\int_{0}^{t} k_i(x,t)F(y_1,y_2,\ldots,y_m) \, dt), \quad i = 1,2,\ldots,m \]  

Equation (3) can be written as:

\[ Y_i(s) = \frac{1}{s^n}(s^{n-1}y_i(0) - s^{n-2}y_i'(0) - \ldots - y_i^{(n-1)}(0)) + \frac{1}{s^n}(F_i(s)) + \frac{1}{s^n}(L(\int_{0}^{t} k_i(x,t)F(y_1,y_2,\ldots,y_m) \, dt)), \quad i = 1,2,\ldots,m \]  

By performing the inverse Laplace transform to equation (4) we get,

\[ y_i(t) = L^{-1}\left(\frac{1}{s^n}(s^{n-1}y_i(0) - s^{n-2}y_i'(0) - \ldots - y_i^{(n-1)}(0)) + \frac{1}{s^n}(F_i(s)) + \frac{1}{s^n}(L(\int_{0}^{t} k_i(x,t)F(y_1,y_2,\ldots,y_m) \, dt))\right), \quad i = 1,2,\ldots,m \]  

According to Adomian decomposition method, we have,

\[ \sum_{n=0}^{\infty} y_{in}(t) = L^{-1}\left(\frac{1}{s^n}(s^{n-1}y_i(0) - s^{n-2}y_i'(0) - \ldots - y_i^{(n-1)}(0)) + \frac{1}{s^n}(F_i(s)) + \frac{1}{s^n}(L(\int_{0}^{t} k_i(x,t)F(\sum_{n=0}^{\infty} y_{1n}, \sum_{n=0}^{\infty} y_{2n},\ldots, \sum_{n=0}^{\infty} y_{mn}) \, dt))\right), \quad i = 1,2,\ldots,m \]  

This implies that

\[
\begin{aligned}
y_i(0) &= L^{-1}\left(\frac{1}{s^n}(s^{n-1}y_i(0) - s^{n-2}y_i'(0) - \ldots - y_i^{(n-1)}(0)) + \frac{1}{s^n}(F_i(s))\right) \\
y_{in} &= L^{-1}\left(\frac{1}{s^n}(L(\int_{0}^{t} k_i(x,t)F(\sum_{n=0}^{\infty} y_{1n}, \sum_{n=0}^{\infty} y_{2n},\ldots, \sum_{n=0}^{\infty} y_{mn}) \, dt))\right), \quad i = 1,2,\ldots,m
\end{aligned}
\]  

2. Laplace Iterative Method (LIM)
Consider the system of Volterra’s integro-differential equations:
\[ y_i^{(n)} = f_i + \int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m) \, dt, \quad i = 1, 2, \ldots, m \tag{8} \]

where \( c_{ij}, j=0,1,2,\ldots,m-1 \) are given constants.

The technique consists at first of implementing Laplace transformation to equation (8), hence

\[ L(y_i^{(n)}) = F_i(s) + \int_0^x \tilde{k}_i(x,t)F(y_1, y_2, \ldots, y_m) \, dt, \quad i = 1, 2, \ldots, m \tag{9} \]

Equation (9) can be simplified as:

\[ \frac{1}{s^n}(F_i(s)) + \int_0^x \tilde{k}_i(x,t)F(y_1, y_2, \ldots, y_m) \, dt, \quad i = 1, 2, \ldots, m \tag{10} \]

Equation (10) can be written as:

\[ Y_i(s) = \frac{1}{s^n}(s^{-n-1}y_i(0) - s^{-n-2}y_i'(0) \ldots - y_i^{(n-1)}(0)) \]

By performing the inverse Laplace transform to equation (11) we get,

\[ y_i(t) = L^{-1}\left(\frac{1}{s^n}(s^{-n-1}y_i(0) - s^{-n-2}y_i'(0) \ldots - y_i^{(n-1)}(0))\right) + \]

\[ L^{-1}\left(\frac{1}{s^n}(F_i(s)) + \int_0^x \tilde{k}_i(x,t)F(y_1, y_2, \ldots, y_m) \, dt\right), \quad i = 1, 2, \ldots, m \tag{12} \]

where

\[ f_i = L^{-1}\left(\frac{1}{s^n}(s^{-n-1}y_i(0) - s^{-n-2}y_i'(0) \ldots - y_i^{(n-1)}(0))\right) \]

\[ A_i(y_1(t), y_2(t), \ldots, y_n(t)) = L^{-1}\left(\frac{1}{s^n}(L\left(\int_0^x \tilde{k}_i(x,t)F(y_1, y_2, \ldots, y_m) \, dt\right))\right), \quad i = 1, 2, \ldots, m \tag{13} \]

We are looking for the solution,

\[ y_i(t) = \sum_{j=0}^{n} y_{ij}(t), \quad i = 1, 2, \ldots, n \tag{14} \]

The nonlinear operators \( A_i \) can be decomposed according to Daftardar-Gejji [14],

\[ A_i(y_1(t), y_2(t), \ldots, y_n(t)) = A_i(y_{10}(t), y_{20}(t), \ldots, y_{n0}(t)) + \]

\[ \sum_{j=0}^{n-1} A_i\left(\sum_{k=0}^{i-j} y_{1k}(t), \ldots, \sum_{k=0}^{i-j} y_{nk}(t)\right) - A_i\left(\sum_{k=0}^{i-1} y_{1k}(t), \ldots, \sum_{k=0}^{i-1} y_{nk}(t)\right) \tag{15} \]

According to equation (14) and equation (15), then equation (12) is equivalent to
\[
\sum_{j=0}^{\infty} y_j(t) = f_i + A_i(y_{10}(t), y_{20}(t), \ldots, y_{n0}(t)) + \\
\sum_{j=0}^{\infty} \left[ A_i \left( \sum_{k=0}^{j-1} y_{ik}(t) \right) \right] - A_i \left( \sum_{k=0}^{j-1} y_{nk}(t) \right)
\]

Hence we have,
\[
y_{i0} = f_i, \quad i = 1, 2, \ldots, m
\]
\[
y_{i1} = A_i(y_{10}(t), y_{20}(t), \ldots, y_{n0}(t)) = L^{-1} \left( \frac{1}{s^n} \left( L \left( \sum_{k=0}^{i} k_i(x,t)F(y_{10}(t), y_{20}(t), \ldots, y_{n0}(t))dt \right) \right) \right), \quad i = 1, 2, \ldots, m
\]
\[
y_{i(j+1)} = L^{-1} \left( \frac{1}{s^n} \left( L \left( \sum_{k=0}^{i-1} k_i(x,t)F(y_{10}(t), y_{20}(t), \ldots, y_{n0}(t))dt \right) \right) \right), \quad i = 1, 2, \ldots, m
\]

For the uniqueness and convergence, see refer to a previous work [14].

3. Illustrative Examples

In this section, two non-linear examples will be given in order to demonstrate the applicability and accuracy of the suggested methods.

Example 1: consider the following equations
\[
y_1^* = 1 - 2 \cos t + \sin t + t^2 - \int_0^t (y_1 + y_2) dt \\
y_2^* = 1 - 2 \sin t - \cos t - \int_0^t (y_2 - y_1) dt
\]

with respect to \( y_1(0) = 1, \ y_2(0) = 0, \ y_1'(0) = 1, \ y_2'(0) = 2 \) (17)

The exact solution is given by an earlier report [15] as \( y_1(t) = t + \cos t, \ y_2(t) = t + \sin t \)

\[
y_{10}(t) = 2t + 2\cos(t) - \sin(t) + \frac{t^2}{12} - 1
\]
\[
y_{11}(t) = 3\sin(t) - \cos(t) - 3t - t^2 + 2 + \frac{t^3}{2} - \frac{t^4}{6} - \frac{t^5}{6} - \frac{t^6}{5} + \frac{t^7}{12} - \frac{t^8}{2520} + 1
\]
\[
y_{12}(t) = 4t - 2\cos(t) - 4\sin(t) - t^2 - 2t^3 + \frac{t^4}{12} + \frac{t^5}{30} - \frac{t^6}{360} + \frac{t^7}{20160} + 2
\]
\[
y_{20}(t) = \frac{t^2}{2} + 2\sin(t) + \cos(t) - 1
\]
\[
y_{21}(t) = t - 3\cos(t) - \sin(t) - \frac{3t^2}{2} + \frac{t^3}{12} + \frac{t^4}{2520} + 3
\]
\[
y_{22}(t) = 2t + 4\cos(t) - 2\sin(t) + 2t^2 - \frac{t^3}{3} - \frac{t^4}{6} + \frac{t^5}{30} + \frac{t^6}{1260} - \frac{t^7}{20160} - \frac{t^8}{907200} - 4
\]
\[
y_1(t) = y_{10}(t) + y_{11}(t) + y_{12}(t)
\]
\[
y_2(t) = y_{20}(t) + y_{21}(t) + y_{22}(t)
\]
Table 1 demonstrates a numerical comparison between the approximate solution of problems (16)-(17) using LADM and LIM with the exact solutions.

<table>
<thead>
<tr>
<th>i</th>
<th>LADM $y_1(t)$</th>
<th>LIM $y_1(t)$</th>
<th>LADM $y_2(t)$</th>
<th>LIM $y_2(t)$</th>
<th>Exact $y_1(t)$</th>
<th>Exact $y_2(t)$</th>
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Table 1 - Numerical comparison between the approximate solutions of problems (16)-(17) using LADM with LIM and the exact solutions.

Figures (1) and (2) represent a numerical comparison between the approximate solutions of $y_1$ and $y_2$ of problems (16) – (17) using LADM and LIM with the exact solutions.

**Figure 1** - Numerical comparison between the approximate solutions of $y_1$ using LADM and LIM with the exact solutions.

**Figure 2** - Numerical comparison between the approximate solutions of $y_2$ using LADM and LIM with the exact solutions.
Example 2: Consider the nonlinear equations

\[
y_1' = 3t^2 \frac{t^3(3t^4 + 7)}{21} - \int_0^t (y_1^2(t) + y_2^2(t))\,dt
\]

\[
y_2' = 1 - \frac{t^2(t^2 - 2)}{4} - \int_0^t (y_2(t) - y_1(t))\,dt
\]

subject to \(y_1(0) = 0, y_2(0) = 0\)

The exact solution is given by \(y_1(t) = t^3, y_2(t) = t\)

\[
y_{10}(t) = \frac{t^3}{36} + \frac{t^4}{12} + t^3
\]

\[
y_{11}(t) = \frac{t^4}{64800} - \frac{t^5}{61152} - \frac{t^6}{4368} - \frac{t^7}{52800} - \frac{t^8}{959616} - \frac{t^9}{432} - \frac{167t^{10}}{10080} - \frac{t^{11}}{90} - \frac{t^{12}}{12}
\]

\[
y_{12}(t) = \frac{t^{10}}{959616} + \frac{t^{11}}{61152} + \frac{t^{12}}{4368} + \frac{t^{13}}{52800} + \frac{t^{14}}{64800} + \frac{t^{15}}{432} + \frac{167t^{16}}{10080} + \frac{t^{17}}{90} + \frac{t^{18}}{12}
\]

\[
y_{20}(t) = \frac{t^3}{6} - \frac{t^4}{20} + \frac{t^5}{6}
\]

\[
y_{21}(t) = \frac{t^{10}}{5040} + \frac{t^{11}}{840} + \frac{t^{12}}{360} + \frac{t^{13}}{24} + \frac{t^{14}}{5040}
\]

\[
y_{22}(t) = \frac{t^4}{6} - \frac{t^5}{840} - \frac{t^6}{360} - \frac{t^7}{24} - \frac{t^{10}}{5040}
\]

\[
y_1(t) = y_{10}(t) + y_{11}(t) + y_{12}(t)
\]

\[
y_2(t) = y_{20}(t) + y_{21}(t) + y_{22}(t)
\]

Table 2 provides a numerical comparison between the approximate solution of problems (18)-(19) using LADM and LIM with the exact solutions.

<table>
<thead>
<tr>
<th>i</th>
<th>LADM y_1(t)</th>
<th>LIM y_1(t)</th>
<th>LADM y_2(t)</th>
<th>LIM y_2(t)</th>
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Table 2 Numerical comparison between the approximate solution of problems (18)-(19) using LADM and LIM with the exact solutions

Figures (3) and (4) represent numerical comparisons between the approximate solutions of \(y_1\) and \(y_2\) of problems (18) – (19) using LADM and LIM with the exact solutions.
4. Conclusions
In this study, the Laplace-Adomian decomposition and Laplace Iterative techniques were successfully employed to investigate a solution of systems of Volterra integro-differential equations. From the numerical results, one can observe that these techniques are powerful and acceptable for solving such types of equations.

References


