Resolution for the two-rowed weyl module in The cases of $(6,5) / (1,0)$ and $(6,5) / (2,0)$

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Abstract
The main purpose of this paper is to study the application of weyl module and resolution in the case skew- shapes $(6, 5) / (1, 0)$ and $(6, 5) / (2, 0)$ by using contracting homotopy and the place polarization.

Key words: weyl module, skew- shape, graded contracting, homotopy.

1: Introduction
Let $R$ be a commutative ring with identity $(1)$ and $F$ be a free $R$-module. In this work we give resolution of weyl modules in the cases of skew partitions $(6,5)/(1,0)$ and $(6,5)/(2,0)$ which are used by D. A. Buchsbaum in the case partition $(2,2)$, Haytham R. Hassan in the case $(3,3)$ and Niran Sabah Jasim in the case $(8,7)$ [1, 2, 3].

Let $Z_{21}$ be the free generator of a divided power algebra
DF $(Z_{21})$ is one generator of the divided element power $Z_{21}$ of degree $k$ of free generator. $Z_{21}$ is $D_{k} F \otimes D_{-k} F$ by place polarization of degree $k$ from place 1 to place 2.

"The graded algebra with identity" $R = DF (Z_{21})$ is the graded module that acts of the graded module $N = \sum D_{k} F \otimes D_{-k} F$ i.e. $N$ is a (graded) left $R$-module;

\[ N \bullet : 0 \rightarrow N_{q} \otimes D_{\delta} \rightarrow N_{q} \otimes N_{q} \rightarrow N_{q} \rightarrow N_{q} \rightarrow \cdots \]

is the weyl module By definition $N_{\bullet}$ is the resolution of $\kappa_{\lambda/\mu}(F)$ where $\kappa_{\lambda/\mu}(F)$

\[ \sum_{k_{1} \geq 0} Z_{21}^{(k_{1}+k)} \otimes Z_{21}^{(k_{2})} \otimes \cdots \otimes Z_{21}^{(k_{i})} \otimes D_{k_{1} + (i+|k|)} F \otimes D_{-k_{1} - (i+|k|)} F \rightarrow \]

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\[
\sum_{k_i \geq 0} Z_{21}^{(k_1)}\sum_{X} Z_{21}^{(k_2)} Z_{21}^{(k_1-1)} D_{b+(t+|k|)} F \otimes D_{q-(t+|k|)} D_{d-1} \rightarrow D_{b} F \otimes D_{q}
\]

... \rightarrow \sum_{k_i \geq 0} Z_{21}^{(k_1)} D_{b+(t+|k|)} F \otimes D_{q-(t+|k|)} F \rightarrow D_{b} F \otimes D_{q}

Where \(|k| = \sum k_i| and d_i is the "boundary operator" \( \partial_X \)

definition of \( S_i \) see [1] and the

\[
S_0 : D_{b} F \otimes D_{q} F \rightarrow \sum_{k_i \geq 0} Z_{21}^{(k_1)} D_{b+(t+|k|)} F \otimes D_{q-(t+|k|)} F
\]

\[
\left[ \begin{array}{c}
\omega \cr \omega
\end{array} \right] \left[ \begin{array}{c}
1 \cr 2 \end{array} \right] \rightarrow \left[ \begin{array}{c}
Z_{21}^{(k_1)} \left( \begin{array}{c}
\omega \\
\omega
\end{array} \right) \left[ \begin{array}{c}
1 \end{array} \right] \\
2(5-\kappa)
\end{array} \right] ; \quad \text{where} \quad \kappa \leq t \quad \text{and} \quad \kappa > t
\]

And for higher dimensions as :

\[
S_{t-1} : \sum_{k_i \geq 0} Z_{21}^{(k_1)} Z_{21}^{(k_2)} Z_{21}^{(k_1-1)} D_{b+(t+|k|)} F \otimes D_{q-(t+|k|)} F
\]

\[
\left[ \begin{array}{c}
\omega \cr \omega
\end{array} \right] \left[ \begin{array}{c}
1 \cr 2 \end{array} \right] \rightarrow \left[ \begin{array}{c}
Z_{21}^{(k_1)} \left( \begin{array}{c}
\omega \\
\omega
\end{array} \right) \left[ \begin{array}{c}
1 \end{array} \right] \\
2(5-\kappa)
\end{array} \right] ; \quad \text{where} \quad \kappa \leq t
\]

As in 1, 4, 5, and 6 that follows, we write the modules of the resolution as \( \mathcal{N}_i \), in which \( i=0,1,\ldots,q-t \); with \( \mathcal{N}_0 = D_{b} F \otimes D_{q} \) and

\[
\mathcal{N}_i = Z_{21}^{(k_1-1)} Z_{21}^{(k_2)} \cdots Z_{21}^{(k_i)} D_{b+i+|k|} F \otimes D_{q-(t+|k|)} F
\]

2. Definitions

(1) Let \( R \) be a commutative ring. A graded \( R \)-algebra is a graded \( R \)-module \( M=\oplus_{i \geq 0} M_i \) together with a "multiplication" homogenous \( m: M \otimes M \rightarrow M \) and a unit \( \eta: R \rightarrow M \).

(2) The Weyl module shape \( \alpha \) is the image of the map \( d_{\alpha}(F) \), and it is denoted by \( l_{\alpha}(F) \).

(3) A contracting homotopy:- \{ \mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots \} is a contracting homotopy if \( \mathcal{S}_{t-1} \partial_X + \partial_X \mathcal{S}_t = \text{id} \).

(4) The divided power algebra \( D F = \sum_{k_i \geq 0} D_i F \) can be defined as the graded commutative algebra generated by element \( \chi^{(i)} \) in degree \( \chi \in F \) and \( I \) is a non negative integer.

3. Application of Weyl modules and resolution in the case skew-shape \((6, 5)/(1, 0)\).

In this section we study the application of Weyl module and resolution in the case \((6, 5)/(1, 0)\) and find the terms of resolution of the skew-shape in the case \((6, 5)/(1, 0)\) as well as the proof of exactness. "Let skew- Shape \((6, 5)/(1, 0)\), in this case we have the following terms of characteristic free resolution".

The characteristic – free resolutions are:

\[
\begin{array}{c}
\mathcal{N}_0 = D_{b} F \otimes D_{q} F \\
\mathcal{N}_1 = Z_{21}^{(1)} D_{b} F \otimes D_{q} F \otimes Z_{21}^{(2)} D_{b} F \otimes D_{q} F \otimes Z_{21}^{(3)} D_{b} F \otimes D_{q} F \\
\mathcal{N}_2 = Z_{21}^{(2)} D_{b} F \otimes D_{q} F \otimes Z_{21}^{(3)} D_{b} F \otimes D_{q} F \\
\mathcal{N}_3 = Z_{21}^{(3)} D_{b} F \otimes D_{q} F \\
\end{array}
\]

Then we have

\[
\mathcal{S}_0 : \mathcal{N}_0 \rightarrow \mathcal{N}_1
\]
\[ S_0 \left( \begin{bmatrix} w \\ \omega \end{bmatrix} \right)^{1(5,2)_{2(5-k)}} = \begin{cases} Z^{(\kappa)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(5+k,2)_{2(5-k)}} & \text{if } \kappa > 2 \\ o & \text{if } \kappa \leq 2 \end{cases} \]

and \( S_1 : N_1 \rightarrow N_2 \) where

\[ S_1 \left( \begin{bmatrix} Z^{(\kappa+1)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+k,2)_{2(1-k-m)}} \right) = \begin{cases} Z^{(\kappa+1, m)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+k+m,2)_{2(1-k-m)}} & \text{if } m = 1,2 \\ o & \text{if } m = 0 \end{cases} \]

and \( S_2 : N_2 \rightarrow N_3 \) where

\[ S_2 \left( \begin{bmatrix} Z^{(\kappa_1+1)}_{21} \times Z^{(\kappa_2)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+|\kappa_1|, 2_{2(1-k-m)})} \right) = \begin{cases} Z^{(\kappa_1+1, \kappa_2, m)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+|\kappa_1|+m,2)_{2(1-k-m)}} & \text{if } m = 1 \\ o & \text{if } m = 0 \end{cases} \]

Since \(|\kappa| = \kappa_1 + \kappa_2\)

"Now we have the following diagram"

\[ \begin{array}{ccccccc}
0 & \xrightarrow{\partial_x} & N_3 & \xleftarrow{id} & N_2 & \xrightarrow{id} & N_1 & \xleftarrow{S_0} & N_0 \\
0 & \xrightarrow{id} & N_3 & \xleftarrow{\partial_x} & N_2 & \xrightarrow{id} & N_1 & \xleftarrow{S_0} & N_0 \\
\end{array} \]

in diagram (5) we can see that

\[ S_0 \partial_x \left( \begin{bmatrix} Z^{(\kappa+1)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+k,2)_{2(1-k-m)}} \right) = S_0 \left( \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(5,2)_{2(1-k-m)}} \right) \]

\[ = S_0 \left( \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(5,2)_{2(1-k-m)}} \right) \]

\[ = S_0 \left( \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(5,2)_{2(1-k-m)}} \right) \]

\[ = \left( \kappa + 1 + m \right) \frac{S_0}{m} \left( \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(5,2)_{2(1-k-m)}} \right) \]

\[ = \left( \kappa + 1 + m \right) \frac{Z^{(\kappa+1+m)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+k+m,2)_{2(1-k-m)}}}{} \]

So that

\[ \partial_x S_1 \left( \begin{bmatrix} Z^{(\kappa+1)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+k,2)_{2(1-k-m)}} \right) = \partial_x \left( \begin{bmatrix} Z^{(\kappa+1)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+k,2)_{2(1-k-m)}} \right) \]

\[ = \partial_x \left( \begin{bmatrix} Z^{(\kappa+1)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+k,2)_{2(1-k-m)}} \right) \]

\[ = - \left( \kappa + 1 + m \right) \frac{Z^{(\kappa+1+m)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+k+m,2)_{2(1-k-m)}}}{} + Z^{(\kappa+1)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+k,2)_{2(1-k-m)}} \]

It is clear that: \( S_0 \partial_x + \partial_x S_1 = \text{id} \).

Now

\[ S_1 \partial_x \left( \begin{bmatrix} Z^{(\kappa_1+1)}_{21} \times Z^{(\kappa_2)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+k,2)_{2(1-k-m)}} \right) \]

\[ = S_1 \left( \frac{|\kappa|}{\kappa_2} \times Z^{(|\kappa|+1)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+k+|\kappa|,2)_{2(1-k-m)}} \right) + Z^{(\kappa+1)}_{21} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+k,2)_{2(1-k-m)}} \]

\[ S_1 \left( \frac{|\kappa|+1}{\kappa_2} \times \begin{bmatrix} w \\ \omega \end{bmatrix}^{1(6+|\kappa|,2)_{2(1-k-m)}} \right) \]
Application of weyl modules and resolution in the case skew-shape $(6, 5)/(2, 0)$.

In this section we study the application of weyl module and resolution in the case $(6, 5)/(2, 0)$ and find the terms of resolution of skew-shape in the case $(6, 5)/(2, 0)$ as well as the proof of exactness.

Let skew-shape $(6, 5)/(2, 0) = (4, 5)$

In this case we have the following terms of characteristic free resolution

$$
\cdots \xrightarrow{S_2} \mathcal{N}_5 \xrightarrow{S_1} \mathcal{N}_2 \xrightarrow{S_0} \mathcal{N}_0 \xrightarrow{id} \mathcal{N} \xrightarrow{id} \mathcal{N} \xrightarrow{id} \mathcal{N} \xrightarrow{id} \mathcal{N} \xrightarrow{id} \cdots
$$

Hence $\{S_0, S_1, S_2\}$ is a contracting homotopy; which implies that the complex $\mathcal{N}$ is exact.

4- Application of weyl modules and resolution in the case skew-shape $(6, 5)/(2, 0)$.

In this section we study the application of weyl module and resolution in the case $(6, 5)/(2, 0)$ and find the terms of resolution of skew-shape in the case $(6, 5)/(2, 0)$ as well as the proof of exactness.
\[
\begin{align*}
\frac{\partial}{\partial \chi} \left( \chi \left( 1^{(6+k)} \right) \right) &= 0 \\
\frac{\partial}{\partial \chi} \left( \chi \left( 1^{(6+k)} \right) \right) &= 0
\end{align*}
\]

In diagram (6) we can see that
\[
\frac{\partial}{\partial \chi} \left( \chi \left( 1^{(6+k)} \right) \right) = \frac{\partial}{\partial \chi} \left( \chi \left( 1^{(6+k)} \right) \right)
\]

From the proposition in diagram (6) we can see that
\[
\frac{\partial}{\partial \chi} + \frac{\partial}{\partial \chi} = \text{id}
\]

Then
\[
\chi \left( 1^{(6+k)} \right) = \chi \left( 1^{(6+k)} \right)
\]

Now
\[
\frac{\partial}{\partial \chi} \left( \chi \left( 1^{(6+k)} \right) \right) = \frac{\partial}{\partial \chi} \left( \chi \left( 1^{(6+k)} \right) \right)
\]

and

\[
(6)
\]
\[ \partial_{\kappa} \square_2 \left( \square_{(\kappa_1+2)} \square_{21} \chi \square_{(\kappa_2)} \square_{21} \chi \left( \square_{(\kappa+2)} \chi \square_{21} \chi \left( \square_{(6+|\kappa|)} \frac{2}{2(2-|\kappa|)-1} \right) \right) \right) \]

\[ = \partial_{\kappa} \left( \square_{(\kappa_1+2)} \square_{21} \chi \square_{(\kappa_2)} \square_{21} \chi \left( \square_{(6+|\kappa|)} \frac{2}{2(2-|\kappa|)-1} \right) \right) \]

\[ = \left( \kappa_2 + \square_{(\kappa_1+2)} \square_{21} \chi \square_{(\kappa_2)} \square_{21} \chi \right) \left( \square_{(6+|\kappa|)} \frac{2}{2(2-|\kappa|)-1} \right) \]

\[ + \square_{(\kappa_2+2)} \square_{21} \chi \square_{(\kappa_2)} \square_{21} \chi \left( \square_{(6+|\kappa|)} \frac{2}{2(2-|\kappa|)-1} \right) \]

In diagram (6) we can see that \( \square_{f} \square_{j} + \square_{g} \square_{2} = \square \)

hence

\[ - \left( \kappa_2 + \square_{(\kappa_1+2)} \square_{21} \chi \square_{(\kappa_2)} \square_{21} \chi \right) \left( \square_{(6+|\kappa|)} \frac{2}{2(2-|\kappa|)-1} \right) \]

\[ + \left( \kappa_2 + \square_{(\kappa_1+2)} \square_{21} \chi \square_{(\kappa_2)} \square_{21} \chi \right) \left( \square_{(6+|\kappa|)} \frac{2}{2(2-|\kappa|)-1} \right) \]

\[ - \kappa_2 \left( \kappa_2 \square_{(\kappa_1+2)} \square_{21} \chi \square_{(\kappa_2)} \square_{21} \chi \right) \left( \square_{(6+|\kappa|)} \frac{2}{2(2-|\kappa|)-1} \right) \]

\[ + \left( \kappa_1 + 2 \right) \left( \kappa_2 \square_{(\kappa_1+2)} \square_{21} \chi \square_{(\kappa_2)} \square_{21} \chi \right) \left( \square_{(6+|\kappa|)} \frac{2}{2(2-|\kappa|)-1} \right) \]

\[ = \left( \kappa_2 \square_{(\kappa_1+2)} \square_{21} \chi \square_{(\kappa_2)} \square_{21} \chi \right) \left( \square_{(6+|\kappa|)} \frac{2}{2(2-|\kappa|)-1} \right) \]

Now we get \( \square_{1} \partial_{\kappa} + \partial_{\kappa} \square_{2} = \text{id} \)

Hence \( \left\{ \square_{f}, \square_{j}, \square_{2} \right\} \) is a contracting homotopy

\[ 0 \xrightarrow{\square_{3}} \square_{1} \xrightarrow{\square_{2}} \rightarrow \rightarrow \rightarrow \] is exact

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References


