T-Stable-extending Modules and Strongly T-stable Extending Modules

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Abstract

In this paper we introduce the notions of t-stable extending and strongly t-stable extending modules. We investigate properties and characterizations of each of these concepts. It is shown that a direct sum of t-stable extending modules is t-stable extending while with certain conditions a direct sum of strongly t-stable extending is strongly t-stable extending. Also, it is proved that under certain condition, a stable submodule of t-stable extending (strongly t-stable extending) inherits the property.

Keywords: extending modules, S-extending module, t-stable extending modules, and strongly t-stable extending modules.

Introduction

Let \( R \) be a ring with unity and \( M \) be a right \( R \)-module. A submodule \( N \) of \( M \) is called essential in \( M \) (\( N \leq_{\text{ess}} M \)) if \( N \cap K = (0) \), \( K \leq M \) implies \( K = (0) \). A submodule \( N \) of \( M \) is called closed in \( M \) if it has no proper essential extension in \( M \), that means if \( N \leq_{\text{ess}} W \), where \( W \leq M \), then \( N = W \) [1], [2]. It is known that for any submodule \( N \) of \( M \), there exists a submodde \( H \) of \( M \), such that \( N \leq_{\text{ess}} H \), hence \( H \) is a closed submodule of \( M \), \( H \) is called a closure of \( N \) [3]. Asgari [4] introduced the notion of t-essential submodule, where a submodule \( N \) of \( M \) is called t-essential (denoted by \( N \leq_{\text{tes}} M \)) if whenever \( W \leq M \), \( N \cap W \leq Z_2(M) \) implies \( W \leq Z_2(M) \), where \( Z_2(M) \) is the second

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singular submodule defined by \( Z \left( \frac{M}{Z(M)} \right) = \frac{Z_2(M)}{Z(M)} \) [1], where \( Z(M) = \{ x \in M : xI = 0 \} \) for some essential ideal of \( R \). Equivalently, \( Z(M) = \{ x \in M : ann(x) \leq ess R \} \) and \( ann(x) = \{ r \in R : xr = 0 \} \) for some t-essential ideal \( I \) of \( R \). \( M \) is called singular (nonsingular) if \( Z(M) = M(Z(M)) = 0 \). Note that \( Z_2(M) = \{ x \in M : xI = 0 \} \) for some t-essential ideal \( I \) of \( R \). \( M \) is called \( Z_2\)-torsion if \( Z_2(M) = M \). Asgari introduced the concept of t-closed submodule where a submodule \( N \) is called t-closed (\( \leq tc \)) if \( N \) has no proper t-essential extension in \( M \) [4]. It is clear that every t-closed submodule is closed, but the converse is not true. However, under the class of nonsingular, the two concepts are equivalent. Asgari [5] stated that for any submodule \( N \) of \( M \), there exists a t-closed submodule \( H \) of \( M \) such that \( N \leq ess H \). \( H \) is called a t-closure of \( N \). A module \( M \) is called extending if for every submodule \( N \) of \( M \) there exists a direct summand \( W (W \leq R M) \) such that \( N \leq ess W \) [6]. Equivalently, \( M \) is an extending module if every closed submodule is a direct summand. As a generalization of extending modules, Asgari [4] introduced the concept of t-extending module, where a module \( M \) is t-extending if every t-closed submodule is a direct summand. Equivalently, \( M \) is t-extending if every submodule of \( M \) is t-essential in a direct summand. The notion of a strongly extending module is introduced in another study [7], which is a subclass of the class of extending module, where an \( R \)-module \( M \) is considered strongly extending if every submodule of \( M \) is essential in a fully invariant direct summand of \( M \) and a submodule \( N \) of \( M \) is called fully invariant if for each \( f \in End(M) \), \( f(N) \leq N \) [8]. A submodule \( N \) of an \( R \)-module \( M \) is called stable if for each \( R \)-homomorphism \( f : N \rightarrow M \), \( f(N) \leq N \) [9]. It is clear that every stable submodule is fully invariant but not conversely. An \( R \)-module \( M \) is fully stable if every submodule of \( M \) is stable [9]. An \( R \)-module \( M \) is called strongly t-extending if every submodule is t-essential in a stable direct summand. Equivalently, \( M \) is strongly t-extending if every t-closed submodule is a fully invariant direct summand [10]. Saad [7] introduced the stable extending (S-extending) modules as a generalization of FI-extending modules. An \( R \)-module \( M \) is called stable extending (S-extending) if every stable submodule of \( M \) is essential in a direct summand of \( M \). A ring \( R \) is left (right) S-extending if \( R \) is S-extending left (right) \( R \)-module and \( M \) is called FI-extending if every fully invariant submodule of \( M \) is essential in a direct summand of \( M \) [11].

In this paper, we introduce the concepts of t-stable extending and strongly t-stable extending modules. The class of t-stable extending modules contains the class of stable extending, and the class of strongly t-stable contains the class of t-stable extending and it is contained in the class of strongly t-extending.

In section two we study t-stable extending modules and their relationships with other related modules. Among other results in this section, we prove that an \( R \)-module \( M \) is a t-stable-extending \( R \)-module if and only if for each stable submodule \( A \) of \( M \), there is a decomposition \( M = M_1 \oplus M_2 \) such that \( A \leq M_1 \) and \( A + M_2 \leq ess M_1 \). An \( R \)-module \( M \) is t-stable extending if and only if for each stable submodule \( K \) of \( M \), there exist \( e = e^2 \in End(E(M)) \) such that \( K \leq ess e(E(M)) \) and \( E(M) \leq M \) where \( E(M) \) is the injective hull of \( M \). Let \( M \) be a stable injective relative to a stable submodule \( X \). If \( M \) is t-stable extending, then so is \( X \).

In section three, we study strongly t-stable extending modules. Many properties are given.

2. T-Stable-extending Modules

In this section we introduce the concept of t-stable extending modules which is a generalization of S-extending modules.

First we give the following definitions.

**Definition 2.1:** An \( R \)-module \( M \) is called t-stable extending if every stable submodule of \( M \) is t-essential in a direct summand. A ring \( R \) is called right t-stable extending if \( R \) is a right t-stable extending \( R \)-module.

Recall that an \( R \)-module is t-uniform if every submodule of \( M \) is t-essential in \( M \) [12]. As a generalization of t-uniform module, we present the following concept.

**Definition 2.2:** An \( R \)-module is called stable-t uniform if every stable submodule of \( M \) is t-essential in \( M \).

**Remarks and Examples 2.3:**

1. It is clear that every S-extending module (or t-extending module) is t-stable extending, for example:
(i) For arbitrary Z-module $M$, $E(M) \oplus Z_2 \oplus Z_8$ is t-extending [4], so it is t-stable extending. Also $Z_2 \oplus Q$ as Z-module is S-extending, so it is t-stable extending.

Recall that an R-module $M$ is called t-continuous if it satisfies the following: $M$ is t-extending, and every submodule of $M$ which contains $Z_2(M)$ and isomorphic to direct summand of $M$ is itself a direct summand [3]. Hence, every t-continuous module is t-stable extending. Hence, we can give the following examples:

(I) By [6, Example 2.6(2)], Let $R$ be a $Z_2$-torsion ring (e.g. $R = \frac{Z}{pZ}$, for a prime number $p$) and set $T = (\begin{matrix} R & R \\ 0 & R \end{matrix})$. $T^2$ t-continuous T-module. It follows that $T^2$ is a t-stable extending module. However, $T^2$ is not stable extending. Hence $T^2$ is not stable extending.

(II) Let $R$ be a ring and $M$ be an $R$-module and $I \leq_{ess} R$. The $R$-module $E(M) \oplus R_I$ is t-continuous [6, Example 2.6(1)], so it is t-stable extending. In particular if $M = Z_p$ as Z-module. Then $Z_p \oplus \frac{Z}{pZ} \approx Z_p \oplus Z_4$ is t-stable.

(2) Let $M$ be a nonsingular R-module. Then $M$ is S-extending if and only if $M$ is t-stable extending. 

Proof: since $M$ is non-singular, then the two concepts essential and t-essential coincide [5]. Hence the two concepts, S-extending and t-stable extending, are equivalent.

(3) If $M$ is a singular module then $M$ is t-stable extending.

Proof: since $M$ is a singular module then $Z_2(M) = M$ and for every submodule $N$ of $M, N + Z_2(M) = N + M = M$, hence $N \leq_{ess} M$ by [5, Prop 1.1]. But $M$ is a direct summand of $M$, so every stable submodule of $M$ is t-essential in a direct summand. Thus $M$ is t-stable extending.

(4) Every FI-t-extending is t-stable-extending where $M$ is FI-t-extending if every fully invariant is t-essential in a direct summand.

Proof: Let $N$ be a stable submodule of $M$. Then $N$ is fully invariant, hence $N$ is t-essential in a direct summand.

(5) The converse of (4) holds if $M$ is FI-quasi-injective, where an R-module $M$ is called FI-quasi-injective if for each fully invariant submodule $N$ of $M$, each R-homomorphism $f: N \to M$ can be extended to an R-endomorphism $g: M \to M$ [7].

Proof: Let $N$ be a fully invariant submodule of $M$. By [7, Proposition 3.1.19] $N$ is stable. Hence by t-stable extending property of $M$, $N$ is t-essential in direct summand. Thus $M$ is a FI-t-extending.

(6) T-stable extending module need not be extending, for example the Z-module $Z_2 \oplus Z_8$ is not extending but it is S-extending by [7, Remarks and Examples 3.1.3(3)] hence it is t-stable extending.

(7) Every stable t-uniform (hence every t-uniform) is t-stable extending.

Proof: Let $N$ be a stable submodule of $M$. Hence $N \leq_{tess} M$. But $M \leq M$, so $N$ is t-essential in a direct summand.

Recall that an $R$-module $M$ is called an S-indecomposable if (0) , $M$ are the only stable direct summand. $M$ is S-extending and S-indecomposable if $M$ is S-uniform. "An R-module $M$ is called stable uniform (shortly, S-uniform) if every stable submodule of $M$ is essential in $M"$ [7]. However we have:

**Proposition 2.4:** If $M$ is t-stable extending and indecomposable, then $M$ is stable t-uniform.

Proof: Let $N$ be a stable submodule in $M$. Then $N \leq_{tess} W$ for some $W \leq M$. Since $M$ is indecomposable, $W = M$. Thus $N \leq_{tess} M$ and so $M$ is a t-uniform.

Note that a stable t-uniform module does not imply indecomposable, for example $Z_6$ as Z-module is stable t-uniform, but $Z_6$ is not indecomposable. Also, $Z_6$ is not S-indecomposable.

**Proposition 2.5:** Let $M$ be an R-module. If $M$ is t-stable extending, then every stable t-closed submodule is a direct summand and the converse holds if every t-closure of stable submodule is stable.

Proof: Let $N$ be a stable t-closed submodule. Since $M$ is t-stable extending, $N \leq_{tess} W$ for some $W \leq M$. Hence $N = W \leq M$, since $N$ is a t-closed. Now if $N$ is a stable submodule of $M$, then $N \leq_{tess} W$, where $W$ is a t-closure of $N$ [5, Lemma 2.3]. By hypothesis, $W$ is stable, and so $W$ is stable t-closed, which implies $W \leq M$. Thus $N$ is t-essential in a direct summand and $M$ is t-stable extending.
**Proposition 2.6:** Let $M$ be an $R$-module which satisfies that the $t$-closure of any submodule is stable. Then $M$ is t-stable extending if and only if $M$ is $t$-extending.

**Proof:** Let $N$ be a $t$-closed of $M$. Hence $N$ is a $t$-closure of $N$ and so by hypothesis, $N$ is stable. But $M$ is $t$-stable extending, so there exists $W \leq M$ such that $N \leq_W W$. Thus $N = W$ because $N$ is $t$-closed and $M$ is $t$-extending.

$\Leftarrow$: If $M$ is $t$-extending, then by Remarks and Examples 2.3(1), $M$ is $t$-stable extending.

**Corollary 2.7:** Let $M$ be a fully stable $R$-module. Then the following statements are equivalent:

1. $M$ is a $t$-stable extending module;
2. $M$ is a $t$-extending module;
3. $M$ is a strongly $t$-extending module.

**Proof:** Since $M$ is a fully stable $R$-module, and the $t$-closure of any submodule of $M$ is stable. Then (1) $\iff$ (2) follows by Proposition 2.6.

(1)$\Rightarrow$(3) Let $N \leq M$. Since $M$ is fully stable, then $N$ is stable. Hence $N$ is $t$-essential in a direct summand $W$. But $W$ is stable in $M$. Then $N$ is $t$-essential in a stable direct summand and so $M$ is strongly $t$-extending.

(3)$\Rightarrow$(2) obvious.

**Proposition 2.8:** Let $M$ be an $R$-module that satisfies that the $t$-closure of any submodule is stable. Then the following statements are equivalent:

1. $M$ is a $t$-stable extending module;
2. Every stable t-closed submodule of $M$ is a direct summand;
3. Every stable submodule is $t$-essential in stable direct summand.

**Proof:** (1)$\Rightarrow$(2) Let $N$ be a stable t-closed submodule. Condition (1) implies $N$ is t-essential in a direct summand $W$. Hence $N = W \leq M$ since $N$ is a $t$-closed.

(2)$\Rightarrow$(3) Let $N$ be a stable submodule in $M$. Then $N$ has a $t$-closure $W$; such that $N \leq W$ and $W$ is a t-closed. But $W$ is stable by hypothesis, so that $W$ is $t$-closed. Then by condition (2) $W \leq M$ and hence $N$ is $t$-essential in a stable direct summand.

(3)$\Rightarrow$(1) clear.

The following are characterizations of the $t$-stable extending modules.

**Theorem 2.9:** An $R$-module $M$ is $t$-stable extending if and only if for each stable submodule $A$ of $M$, there is a decomposition $M = M_1 \oplus M_2$ such that $A \leq M_1$ and $A + M_2 \leq t_{ess} M$.

**Proof:** Suppose $M$ is $t$-stable extending. Let $A$ be a stable submodule of $M$. Then $A \leq t_{ess} M_1 \leq M_1$ and $A + M_2 \leq t_{ess} M = M_1 \oplus M_2$. It follows that $A \leq t_{ess} M_1$.

$\Leftarrow$ Let $A$ be a stable submodule of $M$. By hypothesis, there is a decomposition $M = M_1 \oplus M_2$ with $A \leq M_1$ and $A + M_2 \leq t_{ess} M = M_1 \oplus M_2$. It follows that $A \leq t_{ess} M_1$.

The following is another characterization of $t$-stable extending modules.

**Theorem 2.10:** An $R$-module $M$ is $t$-stable extending if and only if for each stable submodule $K$ of $M$, there exists $e = e^2 \in End(E(M))$ such that $K \leq t_{ess} e(E(M))$ and $e(M) \leq M$ where $E(M)$ is the injective hull of $M$.

**Proof:** Assume $M$ is $t$-stable extending. Let $K$ be a stable submodule of $M$. Then there exists $D \leq D(M)$ such that $K \leq t_{ess} D$ and so there is $H \leq M$ such that $D = D \oplus H$. Hence $E(M) = E(D) \oplus E(H)$. Let $e : E(M) \rightarrow E(D)$ be the projection endomorphism from $E(M)$ onto $E(D)$. Clearly $e^2 = e$ (is idempotent). Thus we have $e(M) \leq (D \oplus H)$. Also, $K \leq t_{ess} D \leq e(M) D$ implies $K \leq t_{ess} e(D) = e(E(M))$.

$\Leftarrow$ Let $K$ be a stable submodule of $M$. By hypothesis, there exists $e \in End(E(M))$, $e^2 = e$ such that $K \leq t_{ess} e(E(M))$ and $e(M) \leq M$. Since $M \leq t_{ess} M$, then $K \cap M \leq t_{ess} e(E(M)) \cap M = e(M)$. It is easy to see that $e(E(M)) \cap M = e(M)$. Also, since $K \cap M = K$, hence $K \leq t_{ess} e(M)$. But $e(M) \leq M$.

**Lemma 2.11:** Let $M = \bigoplus_{i \in I} M_i$. Let $N$ be a stable submodule of $M$. Then $N = \bigoplus_{i \in I} (N \cap M_i)$ where $N \cap M_i$ is stable in $M_i$, $\forall i \in I$. 

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Proof: Let $W$ be a stable submodule. Then $W = \bigoplus_{i \in I} (W \cap M_i)$ by [9, Proposition 4.5] we claim that $N \cap M_i$ is stable in $M_i$, for each $i \in I$. To prove this, let $g: W \cap M_i \rightarrow M_i$ be any $R$-homomorphism. Then $g(W \cap M_i) \subseteq M_i$. Consider the following $W = \bigoplus_{i \in I} (W \cap M_i) \rho$ $W \cap M_i \rightarrow M = \bigoplus_{i \in I} M_i$, where $\rho$ is the natural projection and $i$ is the inclusion mapping. Then $(i \circ g \circ \rho)(W) \subseteq W$ (since $W$ is stable in $M$). But $(i \circ g \circ \rho)(W) = i \circ g(W \cap M_i) = i(g(W \cap M_i)) = g(W \cap M_i)$. Thus $(W \cap M_i)(W) \subseteq W$. From above $g(W \cap M_i) \subseteq M_i$, so we get $g(W \cap M_i) \subseteq W \cap M_i$ and $W \cap M_i$ is a stable submodule of $M_i$, for each $i \in I$.

Theorem 2.12: A direct sum of t-stable extending modules is t-stable extending.

Proof: Suppose that $M = \bigoplus_{i \in I} M_i$, $M_i$ is t-stable extending for each $i \in I$. Let $W$ be a stable submodule of $M$. Then $W = \bigoplus_{i \in I} (W \cap M_i)$ and $W \cap M_i$ is stable in $M_i$ for each $i \in I$ by Lemma 2.11 and so by the t-stable extending property of $M_i$, $W \cap M_i$ is t-essential in a direct summand $N_i$ of $M_i$ for each $i \in I$. Then \( \bigoplus_{i \in I} (W \cap M_i) \leq_{tes} \bigoplus_{i \in I} N_i \) by [5, Corollary 1.3]. Put $N = \bigoplus_{i \in I} N_i$, so $N \leq_{\oplus} M$. Thus $N \leq_{tes} N \leq_{\oplus} M$ and $\square$ is t-stable extending.

Note that any direct sum of extending is S-extending [7, Corollary 3.2.2], hence by Remarks and Examples 2.4(2), it is t-stable extending.

By applying Theorem 2.12, each of $Z_p \bigoplus Z_p \bigoplus Q$ (for each prime number $P$) $Z \bigoplus Z \bigoplus Z \bigoplus Z \bigoplus Z \ldots$ as $Z$-module is t-stable extending. Not that $Z_p \bigoplus Z_p$ and $Z \bigoplus Z \bigoplus Z \ldots$ are not extending. Note that by [7, Corollary 3.2.4] every finitely generated $Z$-module is S-extending, hence it is t-stable extending.

Proposition 2.13: Let $M$ be an $R$-module which satisfies that the t-closure of any submodule is stable. If $M$ is t-stable extending, then every direct summand is t-stable extending.

Proof: Let $N \leq_{\oplus} M$. Since $M$ is t-stable extending, then $M$ is t-extending by Proposition 2.6. Hence $N$ is t-extending by [4, Proposition 2.14(1)]. It follows that $N$ is FI-t-extending and hence by Remarks and Examples 2.3(3), $N$ is t-stable extending.

Corollary 2.14: Let $M$ be a fully stable $R$-module. If $M$ is t-stable extending, then every direct summand is t-stable extending.

Recall that an $R$-module $M$ has the summand intersection property (SIP) if the intersection of two direct summands of $M$ is a direct summand [13]. Since S-extending and t-stable extending are equivalent in the class of nonsingular modules, thus we have every direct summand of t-stable extending module $M$ (where $M$ is nonsingular with SIP) is t-stable extending module. Also, we have by [2, Corollary 3.2.7, and Corollary 3.2.8 and Corollary 3.2.9] the following:

1. Let $M$ be a nonsingular SS-module (that is every direct summand is stable). If $M$ t-stable extending, then every direct summand is t-stable extending.
2. Every direct summand right ideal of a nonsingular t-stable extending commutative ring is t-stable extending.
3. Every direct summand of nonsingular cyclic $Z$-module is t-stable extending.

An $R$-module $M$ is called stable-injective relative to $X$ (simply, $S$-$X$-injective) if for each stable submodule $A$ of $X$, each $R$-homomorphism $f: A \rightarrow M$ can be extended to an $R$-homomorphism $g: X \rightarrow M$ “[7, Definition 3.2.10].

By using the procedure of the proof of Theorem 2.14 [7], we have the following Lemma.

Lemma 2.15: Let $M$ be a stable injective module relative to a stable submodule $X$ of $M$. If $A \subseteq X$ such that $A$ is a stable in $X$, then $A$ is stable in $M$.

Proof: Let $f \in Hom(A, M)$. Since $M$ is stable injective relative to $X$, there exists an $R$-homomorphism $g: X \rightarrow M$ such that $g \circ i = f$ where $i$ is the inclusion mapping from $A$ into $X$. It follows that $g(X) \subseteq X$, since $X$ is stable in $M$. So $g \circ i(A) = g(A) \subseteq X$; that is $g|_A : A \rightarrow X$. But $A$ is stable in $M$, so that $g|_A$ is stable in $A$. Thus $f(A) \subseteq A$ and $A$ is stable in $M$.

Proposition 2.16: Let $M$ be a stable injective relative to a stable submodule $X$. If $M$ t-stable extending, then so is $X$.

Proof: To prove $X$ is t-stable. Let $A$ be a stable submodule of $X$. By Lemma 2.15, $A$ is stable in $M$. Since $M$ is t-stable extending, there exists $D \leq_{\oplus} M$ such that $A \leq_{tes} D$ it follows that $M = D \oplus D'$ for some $D \leq M$ and so $A = X \cap D \leq_{tes} X \cap D \leq_{\oplus} M$ by (5, Corollary 1.3).

3. Strongly t-stable extending modules
In this section, we extend the notion of t-stable extending modules into strongly t-stable extending modules. We study these classes of modules and their relations with some related concepts.

**Definition 3.1:** An R-module $M$ is called strongly t-stable extending if each stable submodule $N$ of $M$. $N$ is t-essential in a stable direct summand.

**Remarks and Examples 3.2:**

1. It is clear that every strongly t-stable extending is t-stable extending.
2. Every strongly t-extending (hence every $Z_2$-torsion) module is strongly t-stable extending. In particular, each of $Z$-module $M = Z_n \oplus Z$ where $n$ is a positive integer is strongly t-extending (see [10, Example 3.3]. Thus $M$ is strongly t-stable extending.
3. The converse of (2) is not true as the following example shows: Let $M = Z_2 \oplus Z$. Let $N$ be a stable submodule of $M$. Then $N = (N \cap Z) \oplus (N \cap Z)$, where $N \cap Z$ is stable in $Z$ by Lemma 2.11. Since the only stable submodules of $Z$ are $Z$, $(0)$, then $N = Z \oplus Z$ or $N = (0) \oplus (0)$ and hence $N \leq_{tes} M \leq^{\oplus} M$. Thus $M$ is a strongly t-stable extending module.
4. Recall that an $R$-module $M$ is called weak duo if every direct summand is fully invariant [14]. Let $M$ be a weak duo. Then $M$ is strongly t-stable extending if and only if $M$ is a t-stable extending module.

**Proof:** $(\Rightarrow)$ It follows by (1)

$(\Leftarrow)$ Let $N$ be a stable submodule of $M$. Then $N \leq_{tes} M \leq^{\oplus} M$. Since $M$ is weak duo, $W$ is a fully invariant in $M$ and then by [7, Lemma 2.1.6] $W$ is stable. Thus $M$ is strongly t-stable extending.

5. Let $M$ be a fully stable module. Then the following are equivalent:
   1. $M$ is t-stable extending;
   2. $M$ is t-extending;
   3. $M$ is strongly t-stable extending;
   4. $M$ is strongly t-extending;
   5. Every stable t-uniform module is strongly t-stable extending.

6. If $M$ is S-indecomposable and $M$ is strongly t-stable extending, then $M$ is a stable t-uniform.

7. If $M$ is S-uniform, then $M$ is strongly t-stable extending and $M$ is S-indecomposable.

8. Let $M$ be an indecomposable module. Then $M$ is strongly t-stable extending if and only if $M$ is t-stable extending.

9. If $M$ is a FI-t-extending, then $M$ is strongly t-stable extending. The converse holds if $M$ is FI-quasi injective.

**Proposition 3.3:** Let $M$ be an $R$-module which satisfies that the t-closure of any submodule is equivalent. Then the following statements are equivalent:

1. $M$ is strongly t-stable extending;
2. $M$ is t-stable extending;
3. $M$ is t-extending;
4. Every stable t-closed is a direct summand;
5. $M$ is strongly t-extending.

**Proof:**

1. $(\Rightarrow)$ Let $N$ be a stable submodule of $N$. Then by definition of strongly t-stable extending, $N$ is t-essential in a fully invariant direct summand. Thus $M$ is strongly t-extending.

2. $(\Rightarrow)$ Since $M$ is t-extending, every t-closed is a direct summand, so it is clear that every stable t-closed is a direct summand.

3. $(\Rightarrow)$ It follows by Proposition 2.8.

4. $(\Rightarrow)$ It follows by Proposition 2.6.
(4) \( \Rightarrow \) (1) Let \( N \) be a stable submodule of \( M \). Then there exists a t-closure of \( N \) say \( W \) such that \( N \leq_{\text{tes}} W \). By hypothesis, \( W \) is stable t-closed of \( M \), hence \( W \leq \otimes M \). Thus \( M \) is strongly t-stable extending.

(5) \( \Rightarrow \) (1) It follows by Remarks and Examples 3.2(2).

(1) \( \Rightarrow \) (5) Let \( N \) be a t-closed of \( M \). Hence \( N \) is a t-closure of \( N \) and so by hypothesis \( N \) is stable.

Since \( M \) is strongly t-stable extending, \( N \leq_{\text{tes}} W \) for some stable direct summand \( W \). It follows that \( N = W \), since \( N \) is t-closed. Thus \( N \) is a stable direct summand and \( M \) is strongly t-extending.

Recall that an \( R \)-module \( M \) is a multiplication module if for each \( N \leq M \), there exists an ideal \( I \) of \( R \) such that \( N = MI \) [15].

**Proposition 3.4:** Let \( M \) be a multiplication t-extending. Then \( M \) is strongly t-stable extending.

**Proof:** Let \( N \) be a stable submodule of \( M \). Since \( M \) is t-stable extending, then there exists \( H \leq \otimes M \) such that \( N \leq_{\text{tes}} H \leq \otimes M \). But \( M \) is a multiplication module implies \( H \) is a fully invariant submodule of \( M \) and so by [7, Lemma 2.1.6], \( H \) is stable. Thus \( M \) is t-essential in stable direct summand of \( M \). Therefore, \( M \) is strongly t-stable extending.

**Corollary 3.5:** Every cyclic t-stable extending module over a commutative ring is strongly t-stable extending.

**Corollary 3.6:** Every commutative t-stable extending ring is strongly t-stable extending.

The following is a characterization of strongly t-stable extending modules.

**Theorem 3.7:** Let \( M \) be an \( R \)-module. \( M \) is strongly t-stable extending if for each stable submodule \( A \) of \( M \), there is a decomposition \( M = M_1 \oplus M_2 \) such that \( A \leq M_1 \) and \( M_2 \) is a stable submodule of \( M \) and \( A + M_2 \leq_{\text{tes}} M_2 \).

**Proof:** Let \( A \) be a stable submodule of \( M \). Since \( M \) is strongly t-stable extending, \( A \leq_{\text{tes}} M_1 \leq \otimes M \) and \( M_1 \) is stable in \( M \). Hence \( M = M_1 \oplus M_2 \) for some \( M_2 \leq M \). Since \( A \leq_{\text{tes}} M_1 \), \( M_2 \leq_{\text{tes}} M_2 \), then \( A + M_2 \leq_{\text{tes}} M_1 \oplus M_2 = M \), by [5, Corollary 1.3].

\[ \iff \] Let \( A \) be a stable submodule of \( M \). By hypothesis, there is a decomposition \( M = M_1 \oplus M_2 \) such that \( A \leq M_1 \), \( M_1 \) is stable in \( M \) and \( A + M_2 \leq_{\text{tes}} M_2 \). Since \( A + M_2 = A \oplus M_2 \leq_{\text{tes}} M_1 \oplus M_2 = M \), then \( A \leq_{\text{tes}} M_1 \). But \( M_1 \) is a stable direct summand of \( M \). Thus \( M \) is strongly t-stable extending.

**Theorem 3.8:** Let \( M = M_1 \oplus M_2 \), where \( M_1 \) and \( M_2 \) are \( R \)-modules, such that \( M \) is an abelian module \( (\text{ann} M_1 R \oplus \text{ann} M_2 R = R) \). If \( M_1 \) and \( M_2 \) are strongly t-stable extending, then \( = f (f_1 \oplus f_2) \) is strongly t-stable extending.

**Proof:** Let \( N \) be a stable submodule of \( M \). By Lemma 2.11, \( N = (N \cap M_1) \oplus (N \cap M_2) \) where \( N \cap M_1 \) is stable in \( M_1 \), \( N \cap M_2 \) is stable in \( M_2 \). Put \( N_1 = (N \cap M_1), N_2 = (N \cap M_2) \). Since \( M_1 \) and \( M_2 \) are strongly t-stable extending, there exist \( W_1 \leq \otimes M_1, W_2 \leq \otimes M_2 \) and \( W_1 \) is stable in \( M_1 \) for \( i = 1, 2 \) and \( N_i \leq_{\text{tes}} W_i \). It follows that \( N_1 \oplus N_2 \leq_{\text{tes}} W_1 \oplus W_2 \) by [5, Corollary 1.3]. Since \( W_1 \leq \otimes M_1, W_2 \leq \otimes M_2 \), then \( W_1 \oplus W_2 \leq \otimes M \). On other hand, \( M \) is abelian (or \( \text{ann} M_1 R \oplus \text{ann} M_2 R = R \)) implies \( \text{Hom}(M_1, M_2) = 0, \text{Hom}(M_2, M_1) = 0 \), by [14, Theorem 4.6]. Hence \( \text{End}(M) \cong \text{End}(M_1) \oplus \text{End}(M_2) \). Since for each \( f \in \text{End}(M) \), \( f = (f_1, f_2) \), \( f_1 \in \text{End}(M_1), f_2 \in \text{End}(M_2) \) and \( f(W_1 \oplus W_2) = f(W_1) \oplus f(W_2) \). But \( W_1 \) and \( W_2 \) are stable in \( M_1, M_2 \) respectively and so that \( f(W_1) \not\subseteq W_1, f(W_2) \not\subseteq W_2 \). Thus \( f(W_1 \oplus W_2) \not\subseteq W_1 \oplus W_2 \), hence \( W_1 \oplus W_2 \) is a fully invariant in \( M \), \( W_1 \oplus W_2 \leq \otimes M \), then [2, Lemma 2.1.6] \( W_1 \oplus W_2 \) is stable in \( M \).

Now we ask the following: Is the property of being strongly t-stable extending inherit to a submodule?

**Definition 3.9:** An \( R \)-module \( M \) is said to be stable-injective if \( M \) is stable-injective to \( N(M) = S-N \)-injective), where \( N(M) \) is any \( R \)-module.

**Theorem 3.10:** Let \( M \) be a stable-injective \( R \)-module. If \( M \) is strongly t-stable extending, then every stable submodule of \( M \) is strongly t-stable extending.

**Proof:** Let \( X \) be a stable submodule of \( M \). To prove \( X \) is strongly t-stable extending, let \( A \) be a stable submodule of \( X \). Since \( M \) is stable-injective, then \( M \) stable-injective relative to \( X \) and hence by Lemma 2.15, \( A \) is strongly t-stable extending and \( A \) is stable in \( M \) imply there
exists a stable direct summand D such that \( A \leq_{\text{tes}} D \leq_{\oplus} M \). Thus \( M = D \oplus D' \) for some \( D' \leq M \). Since \( X \) is stable in \( X = (X \cap D) \oplus (X \cap D') \) where \( X \cap D \) is stable of \( D, X \cap D' \) is stable of \( D' \) by Lemma 2.11. Now \( A \leq_{\text{tes}} D \) implies \( A = X \cap A \leq_{\text{tes}} X \cap D \) by [3,Corollary 1.3]. But \((X \cap D) \leq_{\oplus} X\), so that \( A \leq_{\text{tes}} X \cap A \leq_{\oplus} X \). We claim that \( X \cap D \) is stable in \( X \). Since \( X \cap D \) is stable of \( D \) and \( X \cap D \) is stable in \( D \), then \( X \cap D \) is stable of \( M \) by Lemma 2.15. But \( X \cap D \) is stable in \( M \) and \( X \cap D \subseteq X \) imply \( X \cap D \) is stable in \( X \).

**Proposition 3.11:** Let \( M \) be an \( R \)-module which satisfies that the t-closure of any submodule is stable. If \( M \) is strongly t-stable extending, then every direct summand is strongly t-stable extending.

**Proof:** Let \( W \leq_{\oplus} M \). Since \( M \) satisfies that the t-closure of any submodule is stable, then by (Proposition 3.3) \( M \) is strongly t-extending and so by [8, Theorem 3.5] \( W \) is strongly t-extending. Thus by Remarks and Examples 3.2(2), \( W \) is strongly t-stable extending.

**Corollary 3.12:** Let \( M \) be a fully stable \( R \)-module. If \( M \) is strongly t-stable extending, then every direct summand is strongly t-stable extending.

**References**